Simulation Modelling

Q What is Mathematical Model?

Ans. We define a mathematical mode as a mathematical construction which is designed to study a Particular real-world system or Phenomenon.

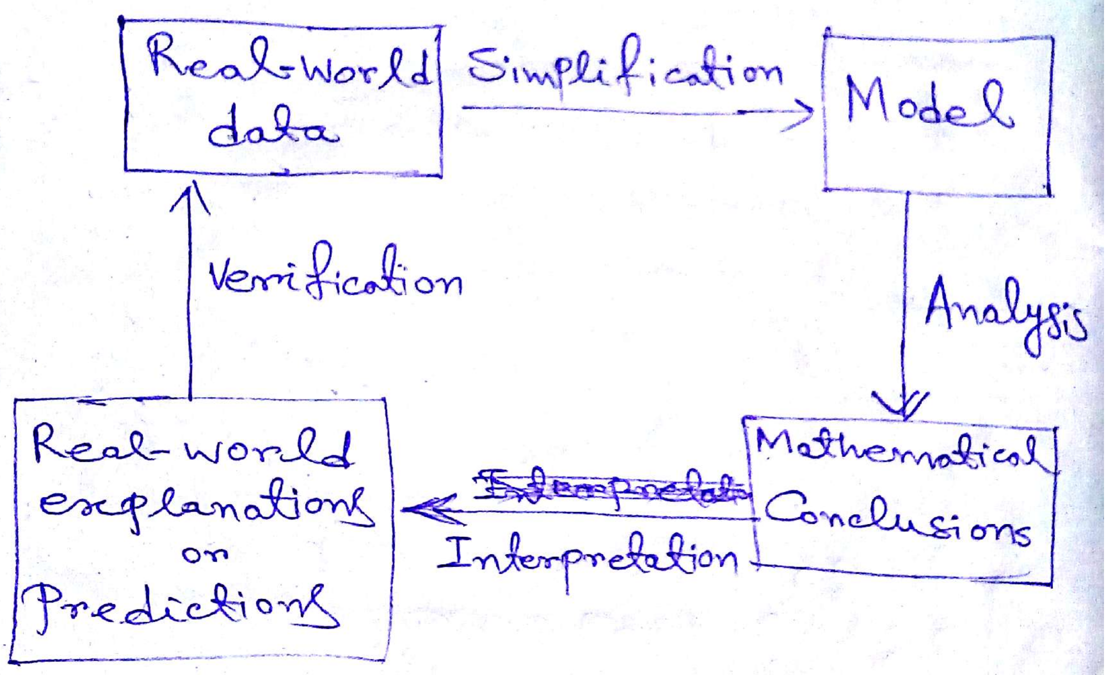


Fig.1: The modelling Process.

Example:- Let us consider a real world problem Such as; Predicting the global effects of the interaction of a population, the use of resources and Pollution.

In such cases we may attempt to replicate the behaviour directly by conducting various experimental trials. Then we collect data from these trials and analyze the data in some way, possibly using statistical techniques or curve-fitting procedures. From the analysis, we can reach certain conclusions.

Note: Using Simulation:- We may attempt to replicate a behaviour on a digital computer — for instance, simulating the global effects of the interactions of population, use of resources, and pollution.
i.e, we may attempt to replicate the behaviour indirectly.

Q Write down the steps for Construction of a mathemati model.

Soln.

Step-1 : Identify the problem.

Step-2 : Make assumptions.

~~Step-3~~ ⓐ Identify and classify the variables.

ⓑ Determine inter-relationships between the variables and Submodels.

Step-3 : Solve the model.

Step-4 : Verify the model.

ⓐ Does it address the Problem ?

ⓑ Does it make common sense ?

ⓒ Test it with real world data

Step-5 : Implement the model

Step-6 ; Maintain the model.

Q What is Simulation Modelling?

Soln Simulation modelling is the Process of creating and analyzing a digital prototype of a Physical model to predict its performance in the real world.

Simulation modelling is used to help designers and engineers understand whether, under what Conditions, and in which ways a Part could fail and what loads it can withstand. It can also help to predict fluid flow and heat transfer patterns.

Q Why use Simulation Modelling?

Soln Simulation modelling solves real world problems safely and efficiently. It provides an important method of analysis which is easily verified, communicated and understood. Across ~~int~~

industries and disciplines, Simulation modelling Provides valuable solutions by giving clear insights into complex systems.

The advantages of simulation modelling are given below:

① Risk free environment.

② Save money and time.

③ Visualisation by animation in 2D / 3D.

④ Increased accuracy.

⑤ Unpredictable data, Uncertainty can be easily represented.

Simulation :-

Simulation is the process of designing a model of a system and conducting experiments with this model for the purpose of understanding the behaviour within the limits imposed by a criterion or set of criterion for the operation of the system.

Advantages of Simulation technique :-

The simulation technique has many advantages. We summarize below a few important ones of them —

i) Simulation models are comparatively flexible and can be modified to adjust the variation in the environment of real situations.

ii) The simulation is an easier technique to use mathematical models. It is quite superior to the mathematical analysis.

iii) Simulation technique has the advantages of being relatively free from complicated mathematics and thus can be easily understood by the operating staff and also by non-technical managers.

Monte-Carlo Simulation :-

The Monte-carlo simulation technique has become so much important part of simulation models that the terms are often are assumed to be synonymous. However, it is only a special technique of simulation. The technique of Monte-carlo involves the selection of random observations within the simulation model.

This technique is restricted for application involving random numbers to solve deterministic and stochastic problems. The principle of technique is replacement of actual statistical universe by another universe described by some assumed probability distribution and then sampling from this theoritical population by means of random numbers.

Infact, this process is the generation of simulated statistics (random variable) that can be explained in sample terms as choosing a random number and substituting this value in standard probability density function to obtain variable or simulated ~~stat~~ statistic.

Steps of Monte-Carlo Simulation :-

The Monte-Carlo Simulation technique Can be applied in the following steps;

Step-1 :- First define the problem by

i) identifying the objectives of the problem.

ii) identifying the main factor having the greatest effect on the objective of the problem.

Step-2 :- Construct the appropriate model by

i) Specifying the variables and Parameter of the model.

ii) formulating the suitable decision rules.

iii) Specifying the number in which time will change.

iv) defining the relationship between the Variable and parameters.

Step-3:- Preparre the model for experimentation by —

i) defining the starting conditions for the simulation and

ii) Specifying the number of runs of simulation to be made.

Step-4:- Using Step-1 to Step-3, test the model by —

i) defining a coding system that will. Correlate the factors defined in Step-1 with the random numbers to be generated for the simulation.

ii) Selecting a random number generator for creating the random numbers to be used in the Simulation.

Step-5:- Summarize and examined the results as obtained.

Step-6:- Evaluate the result of the Simulation.

Step-7:- Formulate the proposals to management on the course of action to be adopted and modify the model, if required.

5

Simulation Modeling

Introduction

In many situations a modeler is unable to construct an analytic (symbolic) model adequately explaining the behavior being observed because of its complexity or the intractability of the proposed explicative model. Yet if it is necessary to make predictions about the behavior, the modeler may conduct experiments (or gather data) to investigate the relationship between the dependent variable(s) and selected values of the independent variable(s) within some range. We constructed empirical models based on collected data in Chapter 4. To collect the data, the modeler may observe the behavior directly. In other instances, the behavior might be duplicated (possibly in a scaled-down version) under controlled conditions, as we will do when predicting the size of craters in Section 14.4.

In some circumstances, it may not be feasible either to observe the behavior directly or to conduct experiments. For instance, consider the service provided by a system of elevators during morning rush hour. After identifying an appropriate problem and defining what is meant by good service, we might suggest some alternative delivery schemes, such as assigning elevators to even and odd floors or using express elevators. Theoretically, each alternative could be tested for some period of time to determine which one provided the best service for particular arrival and destination patterns of the customers. However, such a procedure would probably be very disruptive because it would be necessary to harass the customers constantly as the required statistics were collected. Moreover, the customers would become very confused because the elevator delivery system would keep changing. Another problem concerns testing alternative schemes for controlling automobile traffic in a large city. It would be impractical to constantly change directions of the one-way streets and the distribution of traffic signals to conduct tests.

In still other situations, the system for which alternative procedures need to be tested *may not even exist yet*. An example is the situation of several proposed communications networks, with the problem of determining which is best for a given office building. Still another example is the problem of determining locations of machines in a new industrial plant. The *cost* of conducting experiments may be prohibitive. This is the case when an agency tries to predict the effects of various alternatives for protecting and evacuating the population in case of failure of a nuclear power plant.

In cases where the behavior cannot be explained analytically or data collected directly, the modeler might *simulate* the behavior indirectly in some manner and then test the various alternatives under consideration to estimate how each affects the behavior. Data can then be collected to determine which alternative is best. An example is to determine the drag force on a proposed submarine. Because it is infeasible to build a prototype, we can build a scaled model to simulate the behavior of the actual submarine. Another example of this type of simulation is using a scaled model of a jet airplane in a wind tunnel to estimate the effects of very high speeds for various designs of the aircraft. There is yet another type of simulation, which we will study in this chapter. This **Monte Carlo simulation** is typically accomplished with the aid of a computer.

Suppose we are investigating the service provided by a system of elevators at morning rush hour. In Monte Carlo simulation, the arrival of customers at the elevators during the hour and the destination floors they select need to be replicated. That is, the distribution of arrival times and the distribution of floors desired on the simulated trial must portray a possible rush hour. Moreover, after we have simulated many trials, the daily distribution of arrivals and destinations that occur must mimic the real-world distributions in proper proportions. When we are satisfied that the behavior is adequately duplicated, we can investigate various alternative strategies for operating the elevators. Using a large number of trials, we can gather appropriate statistics, such as the average total delivery time of a customer or the length of the longest queue. These statistics can help determine the best strategy for operating the elevator system.

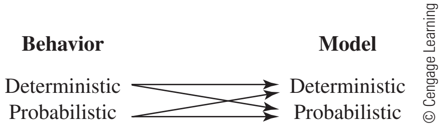
This chapter provides a brief introduction to Monte Carlo simulation. Additional studies in probability and statistics are required to delve into the intricacies of computer simulation and understand its appropriate uses. Nevertheless, you will gain some appreciation of this powerful component of mathematical modeling. Keep in mind that there is a danger in placing too much confidence in the predictions resulting from a simulation, especially if the assumptions inherent in the simulation are not clearly stated. Moreover, the appearance of using large amounts of data and huge amounts of computer time, coupled with the fact the lay people can understand a simulation model and computer output with relative ease, often leads to overconfidence in the results.

When any Monte Carlo simulation is performed, random numbers are used. We discuss how to generate random numbers in Section 5.2. Loosely speaking, a "sequence of random numbers uniformly distributed in an interval m to n" is a set of numbers with no apparent pattern, where each number between m and n can appear with equal likelihood. For example, if you toss a six-sided die 100 times and write down the number showing on the die each time, you will have written down a sequence of 100 random integers approximately uniformly distributed over the interval 1 to 6. Now, suppose that random numbers consisting of six digits can be generated. The tossing of a coin can be duplicated by generating a random number and assigning it a head if the random number is even and a tail if the random number is odd. If this trial is replicated a large number of times, you would expect heads to occur about 50% of the time. However, there is an element of chance involved. It is possible that a run of 100 trials could produce 51 heads and that the next 10 trials could produce all heads (although this is not very likely). Thus, the estimate with 110 trials would actually be worse than the estimate with 100 trials. Processes with an element of chance involved are called **probabilistic**, as opposed to **deterministic**, processes. Monte Carlo simulation is therefore a probabilistic model.

The modeled behavior may be either deterministic or probabilistic. For instance, the area under a curve is deterministic (even though it may be impossible to find it precisely). On the other hand, the time between arrivals of customers at the elevator on a particular day is probabilistic behavior. Referring to Figure 5.1, we see that a deterministic model can be used to approximate either a deterministic or a probabilistic behavior, and likewise, a Monte Carlo simulation can be used to approximate a deterministic behavior (as you will see with

Figure 5.1

The behavior and the model can be either deterministic or probabilistic.



a Monte Carlo approximation to an area under a curve) or a probabilistic one. However, as we would expect, the real power of Monte Carlo simulation lies in modeling a probabilistic behavior.

A principal advantage of Monte Carlo simulation is the relative ease with which it can sometimes be used to approximate very complex probabilistic systems. Additionally, Monte Carlo simulation provides performance estimation over a wide range of conditions rather than a very restricted range as often required by an analytic model. Furthermore, because a particular submodel can be changed rather easily in a Monte Carlo simulation (such as the arrival and destination patterns of customers at the elevators), there is the potential of conducting a sensitivity analysis. Still another advantage is that the modeler has control over the level of detail in a simulation. For example, a very long time frame can be compressed or a small time frame expanded, giving a great advantage over experimental models. Finally, there are very powerful, high-level simulation languages (such as GPSS, GASP, PROLOG, SIMAN, SLAM, and DYNAMO) that eliminate much of the tedious labor in constructing a simulation model.

On the negative side, simulation models are typically expensive to develop and operate. They may require many hours to construct and large amounts of computer time and memory to run. Another disadvantage is that the probabilistic nature of the simulation model limits the conclusions that can be drawn from a particular run unless a sensitivity analysis is conducted. Such an analysis often requires many more runs just to consider a small number of combinations of conditions that can occur in the various submodels. This limitation then forces the modeler to estimate which combination might occur for a particular set of conditions.

5.1 Simulating Deterministic Behavior: Area Under a Curve

In this section we illustrate the use of Monte Carlo simulation to model a deterministic behavior, the area under a curve. We begin by finding an approximate value to the area under a nonnegative curve. Specifically, suppose $y = f(x)$ is some given continuous function satisfying $0 \leq f(x) \leq M$ over the closed interval $a \leq x \leq b$. Here, the number M is simply some constant that *bounds* the function. This situation is depicted in Figure 5.2. Notice that the area we seek is wholly contained within the rectangular region of height M and length $b - a$ (the length of the interval over which f is defined).

Now we select a point $P(x, y)$ at random from within the rectangular region. We will do so by generating two random numbers, x and y, satisfying $a \leq x \leq b$ and $0 \leq y \leq M$, and interpreting them as a point P with coordinates x and y. Once $P(x, y)$ is selected, we ask whether it lies within the region below the curve. That is, does the y-coordinate satisfy $0 \leq y \leq f(x)$? If the answer is yes, then count the point P by adding 1 to some counter.

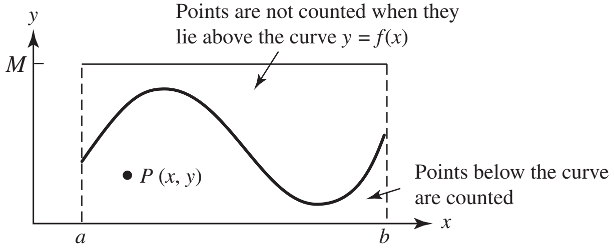


Figure 5.2 The area under the nonnegative curve $y = f(x)$ over $a \le x \le b$ is contained within the rectangle of height M and base length $b - a$.

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Two counters will be necessary: one to count the total points generated and a second to count those points that lie below the curve (Figure 5.2). You can then calculate an approximate value for the area under the curve by the following formula:

$$\frac{\text{area under curve}}{\text{area of rectangle}} \approx \frac{\text{number of points counted below curve}}{\text{total number of random points}}$$

As discussed in the Introduction, the Monte Carlo technique is probabilistic and typically requires a large number of trials before the deviation between the predicted and true values becomes small. A discussion of the number of trials needed to ensure a predetermined level of confidence in the final estimate requires a background in statistics. However, as a general rule, to double the accuracy of the result (i.e., to cut the expected error in half), about four times as many experiments are necessary.

The following algorithm gives the sequence of calculations needed for a general computer simulation of this Monte Carlo technique for finding the area under a curve.

Monte Carlo Area Algorithm

```
Input   Total number $n$ of random points to be generated in the simulation.
Output  AREA = approximate area under the specified curve $y = f(x)$ over the given interval
        $a \le x \le b$, where $0 \le f(x) < M$.
Step 1  Initialize: COUNTER = 0.
Step 2  For $i = 1, 2, \ldots, n$, do Steps 3–5.
    Step 3  Calculate random coordinates $x_i$ and $y_i$ that satisfy $a \le x_i \le b$ and $0 \le y_i < M$.
    Step 4  Calculate $f(x_i)$ for the random $x_i$ coordinate.
    Step 5  If $y_i \le f(x_i)$, then increment the COUNTER by 1. Otherwise, leave COUNTER as is.
Step 6  Calculate AREA = $M(b - a)$ COUNTER/$n$.
Step 7  OUTPUT (AREA)
        STOP
```

Table 5.1 gives the results of several different simulations to obtain the area beneath the curve $y = \cos x$ over the interval $-\pi/2 \le x \le \pi/2$, where $0 \le \cos x < 2$.

The actual area under the curve $y = \cos x$ over the given interval is 2 square units. Note that even with the relatively large number of points generated, the error is significant. For functions of one variable, the Monte Carlo technique is generally not competitive with quadrature techniques that you will learn in numerical analysis. The lack of an error bound and the difficulty in finding an upper bound M are disadvantages as well. Nevertheless, the

Table 5.1 Monte Carlo approximation to the area under the curve $y = \cos x$ over the interval $-\pi/2 \le x \le \pi/2$

Number of points	Approximation to area	Number of points	Approximation to area
100	2.07345	2000	1.94465
200	2.13628	3000	1.97711
300	2.01064	4000	1.99962
400	2.12058	5000	2.01429
500	2.04832	6000	2.02319
600	2.09440	8000	2.00669
700	2.02857	10000	2.00873
800	1.99491	15000	2.00978
900	1.99666	20000	2.01093
1000	1.96664	30000	2.01186

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Monte Carlo technique can be extended to functions of several variables and becomes more practical in that situation.

Volume Under a Surface

Let's consider finding part of the volume of the sphere

$$x^2 + y^2 + z^2 \le 1$$

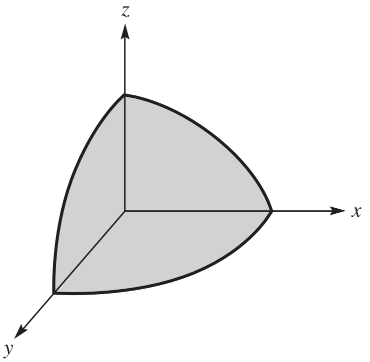
that lies in the first octant, $x > 0$, $y > 0$, $z > 0$ (Figure 5.3).

The methodology to approximate the volume is very similar to that of finding the area under a curve. However, now we will use an approximation for the volume under the surface by the following rule:

$$\frac{\text{volume under surface}}{\text{volume of box}} \approx \frac{\text{number of points counted below surface in 1st octant}}{\text{total number of points}}$$

The following algorithm gives the sequence of calculations required to employ Monte Carlo techniques to find the approximate volume of the region.

Figure 5.3
Volume of a sphere $x^2 + y^2 + z^2 \le 1$ that lies in the first octant, $x > 0$, $y > 0$, $z > 0$



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Monte Carlo Volume Algorithm

```
Input   Total number $n$ of random points to be generated in the simulation.
Output  VOLUME = approximate volume enclosed by the specified function, $z = f(x, y)$ in the
        first octant, $x > 0$, $y > 0$, $z > 0$.
Step 1  Initialize: COUNTER = 0.
Step 2  For $i = 1, 2, \ldots, n$, do Steps 3–5.
  Step 3  Calculate random coordinates $x_i, y_i, z_i$ that satisfy $0 \le x_i \le 1, 0 \le y_i \le 1, 0 \le z_i \le 1$.
          (In general, $a \le x_i \le b, c \le y_i \le d, 0 \le z_i \le M$.)
  Step 4  Calculate $f(x_i, y_i)$ for the random coordinate $(x_i, y_i)$.
  Step 5  If random $z_i \le f(x_i, y_i)$, then increment the COUNTER by 1. Otherwise, leave COUNTER
          as is.
Step 6  Calculate VOLUME = $M(d - c)(b - a)$COUNTER$/n$.
Step 7  OUTPUT (VOLUME)
        STOP
```

Table 5.2 gives the results of several Monte Carlo runs to obtain the approximate volume of

$$x^2 + y^2 + z^2 \le 1$$

that lies in the first octant, $x > 0$, $y > 0$, $z > 0$.

Table 5.2 Monte Carlo approximation to the volume in the first octant under the surface $x^2 + y^2 + z^2 \le 1$

Number of points	Approximate volume
100	0.4700
200	0.5950
300	0.5030
500	0.5140
1,000	0.5180
2,000	0.5120
5,000	0.5180
10,000	0.5234
20,000	0.5242

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The actual volume in the first octant is found to be approximately 0.5236 cubic units $(\pi/6)$. Generally, though not uniformly, the error becomes smaller as the number of points generated increases.

5.1 PROBLEMS

1. Each ticket in a lottery contains a single "hidden" number according to the following scheme: 55% of the tickets contain a 1, 35% contain a 2, and 10% contain a 3. A participant in the lottery wins a prize by obtaining all three numbers 1, 2, and 3. Describe

an experiment that could be used to determine how many tickets you would expect to buy to win a prize.

2. Two record companies, A and B, produce classical music recordings. Label A is a budget label, and 5% of A's new compact discs exhibit significant degrees of warpage. Label B is manufactured under tighter quality control (and consequently more expensive) than A, so only 2% of its compact discs are warped. You purchase one label A and one label B recording at your local store on a regular basis. Describe an experiment that could be used to determine how many times you would expect to make such a purchase before buying two warped compact discs for a given sale.

3. Using Monte Carlo simulation, write an algorithm to calculate an approximation to π by considering the number of random points selected inside the quarter circle

$$Q : x^2 + y^2 = 1, x \geq 0, y \geq 0$$

where the quarter circle is taken to be inside the square

$$S : 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 1$$

Use the equation $\pi/4 = \text{area } Q/\text{area } S$.

4. Use Monte Carlo simulation to approximate the area under the curve $f(x) = \sqrt{x}$, over the interval $\frac{1}{2} \leq x \leq \frac{3}{2}$.

5. Find the area trapped between the two curves $y = x^2$ and $y = 6 - x$ and the x- and y-axes.

6. Using Monte Carlo simulation, write an algorithm to calculate that part of the volume of an ellipsoid

$$\frac{x^2}{2} + \frac{y^2}{4} + \frac{z^2}{8} \leq 16$$

that lies in the first octant, $x > 0$, $y > 0$, $z > 0$.

7. Using Monte Carlo simulation, write an algorithm to calculate the volume trapped between the two paraboloids

$$z = 8 - x^2 - y^2 \quad \text{and} \quad z = x^2 + 3y^2$$

Note that the two paraboloids intersect on the elliptic cylinder

$$x^2 + 2y^2 = 4$$

5.2 Generating Random Numbers

In the previous section, we developed algorithms for Monte Carlo simulations to find areas and volumes. A key ingredient common to these algorithms is the need for random numbers. Random numbers have a variety of applications, including gambling problems, finding an area or volume, and modeling larger complex systems such as large-scale combat operations or air traffic control situations.

In some sense a computer does not really generate random numbers, because computers employ deterministic algorithms. However, we can generate sequences of pseudorandom numbers that, for all practical purposes, may be considered random. There is no single best random number generator or best test to ensure randomness.

There are complete courses of study for random numbers and simulations that cover in depth the methods and tests for pseudorandom number generators. Our purpose here is to introduce a few random number methods that can be utilized to generate sequences of numbers that are nearly random.

Many programming languages, such as Pascal and Basic, and other software (e.g., Minitab, MATLAB, and EXCEL) have built-in random number generators for user convenience.

Middle-Square Method

The middle-square method was developed in 1946 by John Von Neuman, S. Ulm, and N. Metropolis at Los Alamos Laboratories to simulate neutron collisions as part of the Manhattan Project. Their middle-square method works as follows:

1. Start with a four-digit number x_0, called the *seed*.
2. Square it to obtain an eight-digit number (add a leading zero if necessary).
3. Take the middle four digits as the next random number.

Continuing in this manner, we obtain a sequence that appears to be random over the integers from 0 to 9999. These integers can then be scaled to any interval a to b. For example, if we wanted numbers from 0 to 1, we would divide the four-digit numbers by 10,000. Let's illustrate the middle-square method.

Pick a seed, say $x_0 = 2041$, and square it (adding a leading zero) to get 04165681. The middle four digits give the next random number, 1656. Generating 13 random numbers in this way yields

n	0	1	2	3	4	5	6	7	8	9	10	11	12
x_n	2041	1656	7423	1009	0180	0324	1049	1004	80	64	40	16	2

We can use more than 4 digits if we wish, but we always take the middle number of digits equal to the number of digits in the seed. For example, if $x_0 = 653217$ (6 digits), its square 426,692,449,089 has 12 digits. Thus, take the middle 6 digits as the random number, namely, 692449.

The middle-square method is reasonable, but it has a major drawback in its tendency to degenerate to zero (where it will stay forever). With the seed 2041, the random sequence does seem to be approaching zero. How many numbers can be generated until we are almost at zero?

Linear Congruence

The linear congruence method was introduced by D. H. Lehmer in 1951, and a majority of pseudorandom numbers used today are based on this method. One advantage it has over other methods is that seeds can be selected that generate patterns that eventually cycle (we illustrate this concept with an example). However, the length of the cycle is so large that the pattern does not repeat itself on large computers for most applications. The method requires the choice of three integers: a, b, and c. Given some initial seed, say x_0, we generate a sequence by the rule

$$x_{n+1} = (a \times x_n + b) \operatorname{mod}(c)$$

where c is the modulus, a is the multiplier, and b is the increment. The qualifier mod(c) in the equation means to obtain the remainder after dividing the quantity $(a \times x_n + b)$ by c. For example, with $a = 1$, $b = 7$, and $c = 10$,

$$x_{n+1} = (1 \times x_n + 7) \operatorname{mod}(10)$$

means x_{n+1} is the integer remainder upon dividing $x_n + 7$ by 10. Thus, if $x_n = 115$, then $x_{n+1} = \textit{remainder}\ (\frac{122}{10}) = 2$.

Before investigating the linear congruence methodology, we need to discuss **cycling**, which is a major problem that occurs with random numbers. Cycling means the sequence repeats itself, and, although undesirable, it is unavoidable. At some point, all pseudorandom number generators begin to cycle. Let's illustrate cycling with an example.

If we set our seed at $x_0 = 7$, we find $x_1 = (1 \times 7 + 7) \operatorname{mod}(10)$ or $14 \operatorname{mod}(10)$, which is 4. Repeating this same procedure, we obtain the sequence

$$7, 4, 1, 8, 5, 2, 9, 6, 3, 0, 7, 4, \ldots$$

and the original sequence repeats again and again. Note that there is cycling after 10 numbers. The methodology produces a sequence of integers between 0 and $c - 1$ inclusively before cycling (which includes the possible remainders after dividing the integers by c). Cycling is guaranteed with at most c numbers in the random number sequence. Nevertheless, c can be chosen to be very large, and a and b can be chosen in such a way as to obtain a full set of c numbers before cycling begins to occur. Many computers use $c = 2^{31}$ for the large value of c. Again, we can scale the random numbers to obtain a sequence between any limits a and b, as required.

A second problem that can occur with the linear congruence method is lack of statistical independence among the members in the list of random numbers. Any correlations between the nearest neighbors, the next-nearest neighbors, the third-nearest neighbors, and so forth are generally unacceptable. (Because we live in a three-dimensional world, third-nearest neighbor correlations can be particularly damaging in physical applications.) Pseudorandom number sequences can never be completely statistically independent because they are generated by a mathematical formula or algorithm. Nevertheless, the sequence will appear (for practical purposes) independent when it is subjected to certain statistical tests. These concerns are best addressed in a course in statistics.

5.2 PROBLEMS

1. Use the middle-square method to generate
   - a. 10 random numbers using $x_0 = 1009$.
   - b. 20 random numbers using $x_0 = 653217$.
   - c. 15 random numbers using $x_0 = 3043$.
   - d. Comment about the results of each sequence. Was there cycling? Did each sequence degenerate rapidly?
2. Use the linear congruence method to generate
   - a. 10 random numbers using $a = 5$, $b = 1$, and $c = 8$.
   - b. 15 random numbers using $a = 1$, $b = 7$, and $c = 10$.
   - c. 20 random numbers using $a = 5$, $b = 3$, and $c = 16$.
   - d. Comment about the results of each sequence. Was there cycling? If so, when did it occur?

5.2 PROJECTS

1. Complete the requirement for UMAP module 269, "Monte Carlo: The Use of Random Digits to Simulate Experiments," by Dale T. Hoffman. The Monte Carlo technique is presented, explained, and used to find approximate solutions to several realistic problems. Simple experiments are included for student practice.
2. Refer to "Random Numbers" by Mark D. Myerson, UMAP 590. This module discusses methods for generating random numbers and presents tests for determining the randomness of a string of numbers. Complete this module and prepare a short report on testing for randomness.
3. Write a computer program to generate uniformly distributed random integers in the interval $m < x < n$, where m and n are integers, according to the following algorithm:

```
Step 1  Let $d = 2^{31}$ and choose $N$ (the number of random numbers to generate).
Step 2  Choose any seed integer $Y$ such that
            $100000 < Y < 999999$
Step 3  Let $i = 1$.
Step 4  Let $Y = (15625\,Y + 22221)\,\mathrm{mod}(d)$.
Step 5  Let $X_i = m + \mathrm{floor}[(n - m + 1)Y/d]$.
Step 6  Increment $i$ by 1: $i = i + 1$.
Step 7  Go to Step 4 unless $i = N + 1$.
```

Here, floor $[p]$ means the largest integer not exceeding p.

For most choices of Y, the numbers $X_1, X_2, \ldots$ form a sequence of (pseudo)random integers as desired. One possible recommended choice is $Y = 568731$. To generate

chapter 28

QUEUEING THEORY (Waiting Line Models)

INTRODUCTION

In everyday life, it is seen that a number of people arrive at a cinema ticket window. If the people arrive "too frequently" they will have to wait for getting their tickets or sometimes do without it. Under such circumstances, the only alternative is to form a queue, called the ***waiting line,*** in order to maintain a proper discipline. Occasionally, it also happens that the person issuing tickets will have to wait (*i.e.* remains idle) until additional people arrive. Here the arriving people are called the ***customers*** and the person issuing the tickets is called a ***server.***

Another example is represented by letters arriving at a typist's desk. Again, the letters represent the *customers* and the typist represents the *server.* A third example is illustrated by a machine breakdown situation. A broken machine represents a *customer* calling for the service of a repairman. These examples show that the term *customer* may be interpreted in various number of ways. It is also noticed that a service may be performed either by moving the *server* to the *customer* or the *customer* to the *server.*

Thus, it is concluded that waiting lines are not only the lines of human beings but also the aeroplanes seeking to land at busy airport, ships to be unloaded, machine parts to be assembled, cars waiting for traffic lights to turn green, customers waiting for attention in a shop or supermarket, calls arriving at a telephone switch-board, jobs waiting for processing by a computer, or anything else that require work done on and for it are also the examples of costly and critical delay situations. Further, it is also observed that arriving units may form one line and be serviced through only one station (as in a doctor's clinic), may form one line and be served through several stations (as in a barber shop), may form several lines and be served through as many stations (*e.g.* at check out counters of supermarket).

Servers may be in parallel or in series. When in parallel, the arriving customers may form a single queue as shown in *Fig.* 1 (a, b, c) or individual queues in front of each server as is common in big post-offices. Service times may be constant or variable and customers may be served singly or in batches (like passengers boarding a bus).

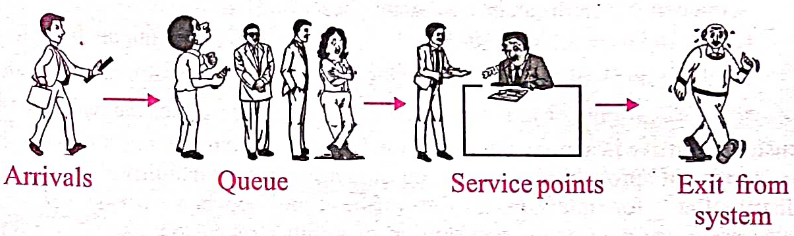


Fig. 1 (a). Queueing system with single queue and single service station.

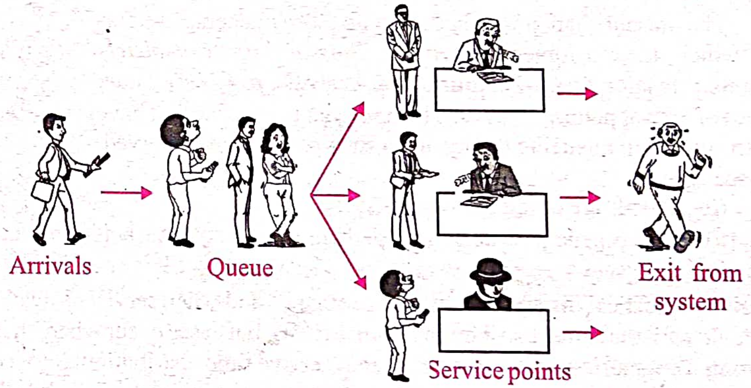


Fig. 1 (b). Queueing system with single queue and several service stations.

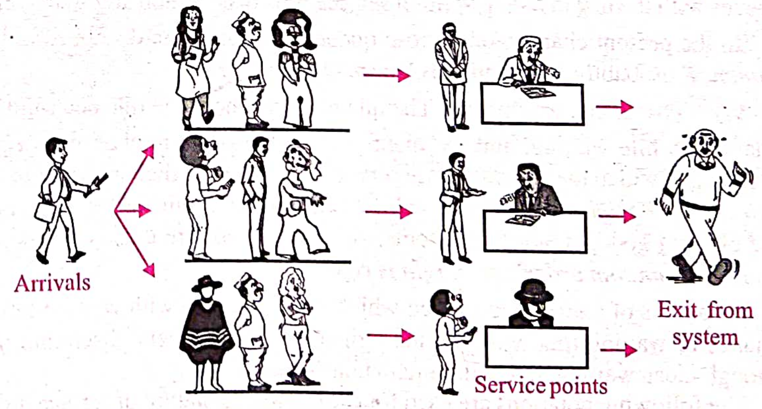


Fig. 1 (c). Queueing system with several queues and several service stations.

Fig. 2 illustrates how a machine shop may be thought of as a system of queues forming in front of a number of service centres, the arrows between the centres indicating possible routes for jobs processed in the shop. Arrivals at a service centre are either new jobs coming into the system or jobs, partially processed, from some other service centre. Departures from a service centre may become the arrivals at another service centre or may leave the system entirely, when processing on these items is complete.

Queueing theory is concerned with the statistical description of the behaviour of queues with finding, *e.g.*, the probability distribution of the number in the queue from which the mean and variance of queue length and the probability distribution of waiting time for a customer, or the distribution of a server's busy periods can be found. In operational research problems involving queues, investigators must measure the existing system to make an objective assessment of its characteristics and must determine how changes may be made to the system, what effects of various kinds of changes in the system's characteristics would be, and whether, in the light of the costs incurred in the systems, changes should be made to it. A model of the queueing system under study must be constructed in this kind of analysis and the results of queueing theory are required to obtain the characteristics of the model and to assess the effects of changes, such as the addition of an extra server or a reduction in mean service time.

Perhaps the most important general fact emerging from the theory is that the degree of congestion in a queueing system (measured by mean wait in the queue or mean queue length) is very much dependent on the amount of irregularity in the system. Thus congestion depends not just on mean rates at which customers arrive and are served and may be reduced without altering mean rates by regularizing arrivals or service times, or both where this can be achieved.

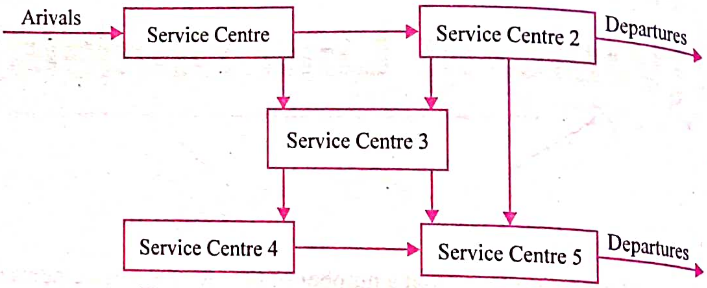


Fig. 2. A machine shop as a complex queue.

QUEUEING SYSTEM

A queueing system can be completely described by

(*a*) the input (or arrival pattern)

(*b*) the service mechanism (or service pattern)

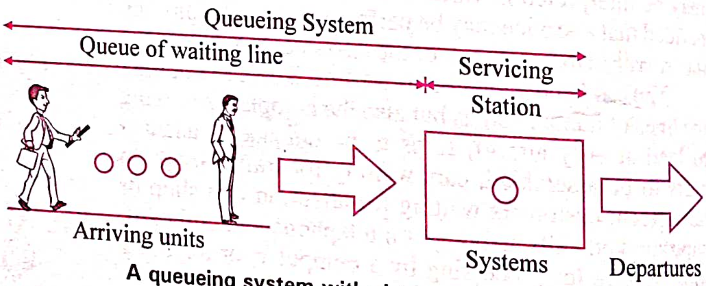
(*c*) the 'queue discipline

(*d*) customer's behaviour

(*e*) Size of a population

(*f*) Maximum length of queue.

(*a*) **The input (or arrival pattern) :** The input describes the way in which the customers arrive and join the system. Generally, the customers arrive in a more or less random fashion which is not worth making the prediction. Thus, the arrival pattern can best be described in terms of probabilities and consequently the probability distribution for inter-arrival times (the time between two successive arrivals) or the distribution of number of customers arriving in unit time must be defined.



A queueing system with single service station.

The present chapter is only dealt with those queueing systems in which the customers arrive in *'Poisson'* or *'Completely random'* fashion (**see Distribution of Arrivals, *p.* QT/4**). Other types of arrival patterns may also be observed in practice that have been studied in queueing theory. Two such patterns are observed, where

(*i*) arrivals are of regular intervals;

(*ii*) there is general distribution (perhaps normal) of time between successive arrivals.

(*b*) **The service mechanism (or service pattern) :** It is specified when it is known how many customers can be served at a time, what the statistical distribution of service time is, and when service is available. It is true in most situations that service time is a random variable with the same distribution for all arrivals, but cases occur where there are clearly two or more classes of customers (*e.g.* machines waiting for repairing) each with a different service time distribution. Service time may be constant or a random variable. Distributions of service time which are important in practice are ***'negative exponential distribution'*** and the related **'*Erlang (Gamma) distribution*'**. Queues with the negative exponential service time distribution are studied in the following sections.

In the present chapter, only those queueing systems are discussed in which the service time follows the *'Exponential and Erlang (Gamma)'* probability distributions (*see p.* QT/4—QT/10).

(*c*) **The queue discipline :** The queue discipline is the rule determining the formation of the queue, the manner of the customer's behaviour while waiting, and the manner in which they are chosen for service. The simplest discipline is ***"first come, first served"***, according to which the customers are served in the order of their arrival. For example, such type of queue discipline is observed *at a ration shop, at cinema ticket windows, at railway stations*, etc. If the order is reversed, we have the ***"last come, first served"*** discipline, as in the case of a big godown where the items which come last are taken out first. An extremely difficult queue discipline to handle might be ***"service in random order"*** or ***"might is right"***.

Properties of a queueing system which are concerned with waiting times, in general, depend on queue discipline. For example, the variance of waiting time will be much greater with the queue discipline ***'first come, last served'*** than with ***'first come, first served'***, although mean waiting time will remain unaffected.

The following notations are used for describing the nature of service discipline.

FIFO → *First In, First Out* or *FCFS* → *First Come, First Served.*
LIFO → *Last In, First Out* or *FILO* → *First In, Last Out.*
SIRO → *Service in Random Order.*

This chapter shall be concerned only with the customers which are served in the order in which they arrive at the service facility, that is, ***'first come, first served'*** discipline.

(*d*) **Customer's behaviour :** The customers generally behave in four ways :

- (*i*) **Balking.** A customer may leave the queue because the queue is too long and he has no time to wait, or there is not sufficient waiting space.
- (*ii*) **Reneging.** This occurs when a waiting customer leaves the queue due to impatience.
- (*iii*) **Priorities.** In certain applications, some customers are served before others regardless of their order of arrival. These customers have *priority* over others.
- (*iv*) **Jockeying.** Customers may *jockey* from one waiting line to another. It may be seen that this occurs in the supermarket.

(*e*) **Size of a population :** The collection of potential customers may be very large or of a moderate size . In a railway booking counter the total number of potential passengers is so large that although ***theoretically finite*** it can be ***regarded as infinity*** for all practical purposes. ***The assumption of infinite population is very convenient for analysing a queuing model.*** However, this assumption is not valid where the customer group is represented by few machines in workshop that require operator facility from time to time. If the population size is finite then the analysis of queueing model becomes more involved.

(*f*) **Maximum length of a queue :** Sometimes only a finite number of customers are allowed to stay in the system although the total number of customers in the population may or may not be finite. For example, a doctor may have appointments with k patients in a day. If the number of patients asking for appointment exceeds k, they are not allowed to join the queue. Thus, although the size of the population is infinite, the maximum number permissible in the system is k.

Q. 1. What do you understand by queue discipline and service process? **[Madras (MBA) 2006]**

2. Explain briefly the main characteristics of queueing system. **[Delhi (MBA) 2005; Annamalai (MBA) 2002]**

3. Describe the fundamental components of a queueing process and give suitable examples. **[C.A. (Nov) 1992]**

4. List the factors that constitute the basic elements of a queueing model. For each of these enumerate the alternatives posible. Represent this diagramatically to cover all possible implimentations of a queueing model. **[IGNOU 1999 (Dec.)]**

5. What is queueing theory? In what type of situations it can be applied successively? Discuss giving examples. **[Delhi (MBA) 2009]**

6. Describe the fundamental components of a queueing system and give suitable example. **[GBTU (MBA II Sem.) 2011]**

7. Discuss the essential features of queueing system. **[JNTU (MBA II Sem.) 2011]**

8. What is Queuing Theory? Also explain Queueing System. **[GBTU (MBA II Sem.) 2012]**

QUEUEING PROBLEM

In a specified queueing system, the problem is to determine the following :

(*a*) **Probability distribution of queue length :** When the nature of probability distributions of the arrival and service patterns is given, the probability distribution of queue length can be obtained. Further, we can also estimate the probability that there is no queue.

(*b*) **Probability distribution of waiting time of customers :** We can find the time spent by a customer in the queue before the commencement of his service which is called his ***waiting time.*** The **total time** spent by him in the system is the waiting time plus service time.

(*c*) **The busy period distribution :** We can estimate the probability distribution of busy periods. If we suppose that the server is free initially and customer arrives, he will be served immediately. During his service time, some more customers will arrive and will be served in their turn. This process will continue in this way until no customer is left unserved and the server becomes free again. Whenever this happens, we say that a ***busy period*** has just ended. On the other hand, during **idle periods** no customer is present in the system. A busy period and the idle period following it together constitute a ***busy cycle.*** The study of the busy period is of great interest in cases where technical features of the server and his capacity for continuous operations must be taken into account.

TRANSIENT AND STEADY STATES

Queueing theory analysis involves the study of a system's behaviour over time. ***A system is said to be in "transient state" when its operating characteristics (behaviour) are dependent on time.*** This usually occurs at the early stages of the operation of the system where its behaviour is still dependent on the initial conditions. However, since we are mostly interested in the *"long run"* behaviour of the system, mainly the attention has been paid toward *"steady state''* results.

A steady state condition is said to prevail when the behaviour of the system becomes independent of time. Let $P_n(t)$ denote the probability that there are n units in the system at time t. In fact, the change of $P_n(t)$ with respect to t is described by the derivative $[dP_n(t)/dt]$ or $P_n'(t)$. Then the queueing system is said to become *'stable'* eventually, in the sense that the probability $P_n(t)$ is independent of time, that is, remains the same as time passes ($t \to \infty$). Mathematically, in steady state

$$\lim_{t\to\infty} P_n(t) = P_n \text{ (independent of } t\text{)}$$

$$\to \lim_{t\to\infty} \frac{dP_n(t)}{dt} = \frac{dP_n}{dt} \to \lim_{t\to\infty} P_n'(t) = 0.$$

In some situations, if the arrival rate of the system is larger than its service rate, a steady state cannot be reached regardless of the length of the elapsed time. In fact, in this case the queue length will increase with time and theoretically it could build upto infinity. Such case is called the ***"explosive state".*** (If $\lambda > \mu$, no steady state.)

In this chapter, only the steady state analysis will be considered. We shall not treat the *'transient'* and *'explosive'* states.

Overview of Optimization Modeling

The basic model is

$$\text{optimize } f_j(x) \text{ for } j \in J.$$

Subject to,

$$g_i(x) \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b_i \ \forall\, i \in I.$$

Explanations

i) To optimize means to maximize or to minimize.

ii) The subscript j indicates that there may be one or more functions to optimize. The functions are distinguished by the integer subscripts that belong to the finite set J.

iii) For the set of functions $f_j(x)$, we have to find the optimal solution x_0.

iv) The various components of the vector x are called the decision variables of the model.

v) The functions $f_j(x)$ are called the Objective functions.

vi) The term "Subject to" indicates the side conditions of the model which are compatible for the model.

vii) Side conditions are called Constraints. The integer subscript i indicates that there may be one or more constraint relationships that must be satisfied.

viii) A constraint may be an equality or an inequality.

ix) Each constant b_i represents the level that the associated constraint function $g_i(X)$ must achieve. This b_i is called the right hand side in the model.

x) Finally, we have to find the solution vector X_0 that must optimize each of the objective functions $f_j(X)$ and simultaneously satisfy each constraint relationship.

Classification of Some Optimization Problem

The different classifications of optimization problems are to describe certain mathematical characteristics possessed by the problem under investigation.

Optimization Problem

- Unconstrained Optimization Problem.
- Constrained Optimization Problem.
- Linear Programming Problem (LPP).
- Non-linear optimization Problem (NLPP)
- Multiobjective Problem/Goal Programming Problem (GPP).
- Dynamic Programming Problem (DPP).
- Stochastic Programming Problem (SPP).
- Integer Programming Problem (IPP).
- Mixed Integer Programming Problem (MIPP).

Properties of LPP :-

An optimization problem is said to be a linear programming problem (LPP) if it satisfies the following properties :

i) There is a unique objective function.

ii) Whenever a decision variable appears in either the objective function or one of the constraint functions, it must appear only as a power term with an exponent of 1, possibly multipeied by a constant.

iii) No term in the objective function or in any of the constraints can contain products of the decision variables.

iv) The coefficients of the decision variables in the objective functions & each constraint are constant.

v) The decision variables are permitted to assume fractional as well as integer values.

Goal Programming Problem :- Goal Programming is an approach used for solving a multi-objective optimization problem that balences a trade-off in conflicting objectives. It can be thought of as an extension or generalization of linear programming to handle multiple, normally conflicting objective measures. Each of these measures gives a goal or target value to be achieved. Unwanted deviations from this set of target values are minimized by an achievement function.

The general GP problem ~~Integer Programming Problem:~~ can be stated as follows:

$$\text{Minimize } Z = \sum_{i=1}^{m} w_i (d_i^- + d_i^+)$$

$$\text{Subject to, } \sum_{j=1}^{n} a_{ij} x_j + d_i^- - d_i^+ = b_i, \; i=1,2,.., m$$

$$\text{and} \quad x_j, d_i^-, d_i^+ \geq 0, \text{ for all } i,j$$

Where m goals are expressed by an m-component column b_i, a_{ij} represents the coefficient for the j-th decision variable in the i-th constraint,

x_j represents a decision variable, w_i represents the weights of each goal and d_i^-, d_i^+ are deviational variables representing the amount of under-achievement and over-achievement of i-th goal respectively.

Integer Programming Problem:— A linear Programming problem in which some or all variables $x_1, x_2, \ldots, x_n$ are permitted to take the integral values (whole numbers), is referred as an integer programming problem (IPP). Mathematical model is as follows:

Optimize : $\sum_{j=1}^{n} c_j x_j$

Subject to, $\sum_j a_{ij} x_j = b_i$, $i = 1, 2, \ldots, m$

and $x_j \geqslant 0$, $j = 1, 2, \ldots, n$.

and x_j integer valued for $j = 1, 2, \ldots, p \leq n$.

An IPP is termed as pure IPP if the all variables are restricted to take only integral values, i.e., $p = n$, otherwise if $p < n$, i.e., if some (say p) variables are restricted to take only integer values and $(n-p)$ remaining variables are free to take any non-negative values, then the problem is called a mixed IPP.

Non linear Programming Problem :- The mathematical formulation of general non-linear programming Problem may be expressed as follows:

Max (or Min) $Z = C(x_1, x_2, .., x_n)$

Subject to,

$a_1(x_1, x_2, .., x_n) \{\leq, = \text{ or } \geq\} b_1$

$a_2(x_1, x_2, .., x_n) \{\leq, = \text{ or } \geq\} b_2$

$- - - - - - - - - - - - -$

$a_m(x_1, x_2, .., x_n) \{\leq, = \text{ or } \geq\} b_m$

and $x_j \geq 0$, $j = 1, 2, .., n$.

where either $C(x_1, x_2, .., x_n)$ or some $a_i(x_1, x_2, .., x_n)$, $i = 1, 2, .., m$. or both are non-linear.

*No general algorithms are available for dealing with non-linear models. The reason for this is mainly the irregular behaviour of the non-linear functions. Although, a large number of algorithms have been developed for the solution of non-linear programming problem, even then there is a need of developing a more efficient solution procedure.

Stochastic Programming Problem:- Stochastic Programming is a framework for modeling optimization problems that involve uncertainty. A stochastic program is an optimization problem in which some or all problem parameters are uncertain, but follow known probability distributions. This framework contrasts with deterministic optimization, in which all problem parameters are assumed to be known exactly.

The goal of stochastic programming is to find a decision which both optimizes some criteria chosen by the decision maker, and appropriately accounts for the uncertainty of the problem parameters. Because many real world decisions involve uncertainty, stochastic programming has found applications in a broad range of areas ranging from finance to transportation to energy optimization.

Concept of dynamic Programming Problem (DPP) :-

"Consider an optimal sub-division Problem where a positive quantity b is to be divided into n parts. The object is to determine the optimum sub-division of b in order to maximize the product of n parts."

The dynamic programming approach ~~remove these~~ breaks the problem into smaller sub problems, and each sub problem is referred to as a stage. A stage signifies a portion of the decision problem for which a separate decision can be taken. The resulting decision will also be meaningful if it is optimal for the stage it represents and can be used directly as a part of the optimal solution to the problem. In general, number of stages in a problem may be finite or infinite.

Solution Procedure of Solving L.P.P. using Graphical Method.

Step-1:- Consider each inequality constraint as an equation.

Step-2:- Plot each equation on the graph, as each one will geometrically represent a straight line.

Step-3:- Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality constraint corresponding to that line is '≤', then the ~~line~~ region below the line lying in the first quadrant (due to non-negative of variables) is shaded. For the inequality constraint with '≥' sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

Step-4:- Choose the convenient value of Z (say $=0$) and plot the objective function line.

Step-5:- Pull the objective function line untill the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step-6 :- Read the co-ordinates of the extreme poir Selected in step 5, and find the maximum or minimum (as the case may be) value of Z.

Areas of application of linear programming :-

LPP is used to solve problems of procurement of raw materials in changing situations, production planning, assembly line balencing and many other problems of operation management. In the field of marketing LPP is used to solve problems of market mix, location of warehouses, blending and many other day to day problems associated with marketing. In the field of finance LPP is used in financing, profit planning and investment.

Also, LPP is used extensively in Government and public-services, diet-mix in hospitals, educational planning, air line and crew-scheduling and in food shipping plan.

Advantages of LPP

The advantages of LPP may be outlined as follows:

1) LPP indicates how a decision maker can employ his productive factors most effectively by choosing and allocating these resources.

2) The quality of decisions may also be improved by LPP. The user of this technique becomes more objective and less subjective.

3) LPP provides necessary modification of its mathematical solution for the sake of convenience to the decision maker.

4) In production processes, bottleneck problem solving is a very significant advantage of this technique.

Limitations of LPP

Some limitations are associated with linear programming techniques. These are stated below:

1) In real life situations concerning business and industrial problems constraints are not linearly treated to variables.

2) In linear programming technique there is no guarantee of getting integer valued solutions.

3) Linear programming model does not take into consideration the effect of time and uncertainty.

4) Sometimes large scale problems cannot be solved with linear programming techniques even when the computer facility is available.

5) Parameters appearing in the model are assumed to be constant. But, in real life situations they are neither constant nor deterministic.

6) Linear programming deals with only single objective, whereas in real life situations problems come across with multi objectives.

Fundamental theorem of LPP :-

If the LPP admits of an optimal solution, then the optimal solution will coincide with at least one basic feasible solution of the problem.

Simplex Algorithm :-

The simplex algorithm is an iterative procedure for solving LP problems. It consists of

i) having a trial basic feasible solution to constraint equations,

ii) testing whether it is an optimal solution or not,

iii) Improving the first trial solution by a set of rules, and repeating the process till and optimal solution is obtained.

Computational procedure of simplex method:-

Simplex method is an iterative procedure involving the following steps:-

Step-1:- If the problem is one of minimization, convert it to a maximization problem by considering $-Z$, instead of Z, using the fact $\min Z = +\max(-Z)$

Step-2:- We check up all b_i's for non-negativity. If some of the b_i's are negative, multipey the corresponding constraint through by (-1) in order to ensure all $b_i \geq 0$.

Step-3:- We change the inequalities to equations by adding slack and Surplus variables (if ~~any~~. necessary).

Step-4:- We add artificial variables to those constraints with ($\geq$) or ($=$) sign in order to get the identity basis matrix.

Step-5:- We now construct the starting simplex table. From this table, the initial basic feasible solution can be read off.

$c_j \rightarrow \quad c_1 \; c_2 \; c_3 \; -- c_k --- c_{m+n}$

Basic Variables	C_B	X_B	$x_1 \; x_2 \; x_3 \; -- x_k --- x_{m+n}$	Min Ratio
------	---	---	-- --- --- --- ---	---
	$Z = C_B X_B$		$\Delta_1 \; \Delta_2 \; \Delta_3 -- \Delta_k --- \Delta_{m+n}$	$\leftarrow \Delta_j$

Form of Simplex table.

Step-6:- We obtain the values of Δ_j by the formulas $\Delta_j = Z_j - C_j = C_B X_B - c_j$, and examine the values of Δ_j. There will be three mutually exclusive and collectively exhaustive possibilities.

i) All $\Delta_j \geq 0$. In this case, the basic feasible solution under test will be optimal.

ii) Some $\Delta_j < 0$, and for at least one of the corresponding X_j all $x_{rj} \leq 0$. In this case, the solution will be unbounded.

iii) Some $\Delta_j \leq 0$, and all the corresponding X_j's have at least one $x_{rj} > 0$. In this case, there is no end of the road. So, further improvement is possible.

Step-7:- Further improvement is done by replacing one of the vectors at present in the basis matrix by that one outside the basis. We use the following rules to select such a vector:

i) To select "incoming vector". We find such value of k for which $\Delta_K = \min \Delta_j$. Then the vector coming into the basis matrix will be X_K.

ii) To select "Outgoing vector". The vector going out of the basis matrix will be β_r, if we determine the suffix r by the minimum ratio rule $\frac{x_{Br}}{x_{rK}} =$ for predetermined value of K.

Step-8 :- We now construct the next improvement table by using the Simple matrix transformation rules.

Step-9 :- Now return to Step-6, then go the Steps 8 and 9, if necessary. This Process is repeated till we reach the disired conclusion.

Sensitivity Analysis :-

Sensitivity analysis is the Study of how changes in the coefficients of a linear program affect the optimal Solution. Using sensitivity analysis, we can answer questions such as the following :

1) How will a change in a coefficient of the ~~obt~~ objective function affect the optimal Solution?

2) How will a change in the right hand side value for a constraint affect the optimal Solution?

Since sensitivity analysis is concerned with how the above changes affect the optimal solution, the analysis does not begin until the optimal solution of the original LPP has been ~~obtained~~ obtained. For this reason, sensitivity analysis is often referred to as post-optimality analysis.

Significance of sensitivity analysis :-

The primary reason that sensitivity analysis is important to decision makers is that real world problems exist in a dynamic environment. Prices of raw materials change, demand fluctuates, companies purchases new machinary to replace the old, global labour markets cause changes in production costs, employee turnover occurs and so on. If a linear programming model has been used in such an environment, we can expect some of the coefficients to change over time. A manager would like to determine how such changes affect the optimal solution to the original LPP. Sensitivity analysis provides the information needed to respond to such changes without requiring the complete solution of a revised linear program.

GRAPHICAL METHOD

1. Solution Procedure

Simple linear programming problems of two decision variables can be easily solved by *graphical method.*

The outlines of graphical procedure are as follows :

Step 1. Consider each inequality–constraint as an equation.

Step 2. Plot each equation on the graph, as each one will geometrically represent a straight line.

Step 3. Shade the feasible region. Every point on the line will satisfy the equation of the line. If the inequality-constraint corresponding to that line is '≤', then the region *below* the line *lying* in the first quadrant (due to non-negativity of variables) is shaded. For the inequality-constraint with '≥' sign, the region *above* the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the *feasible region.*

Step 4. Choose the convenient value of z (say = 0) and plot the objective function line.

Step 5. Pull the objective function line until the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passing through at least one corner of the feasible region. In the minimization case, this line will stop nearest to the origin and passing through at least one corner of the feasible region.

Step 6. Read the coordinates of the extreme point(s) selected in *Step 5*, and find the maximum or mininum (as the case may be) value of z. The following examples will make the graphical procedure clear.

Q. 1. What is meant by linear programming problem ? Give brief description of the problem with illustrations. How the same can be solved graphically. What are the basic characteristics of a linear programming problem ?

2. Explain briefly the graphical method of solving linear programming problems. State its advantages and limitations.
[JNTU (IV B. Tech.) I Sem. Feb. 2007]

3. Write the algorithm of graphical solution for LP models.
[JNTU (MCA III) 2004]

4. Show on a graph the following :
(i) Unbounded solution space (ii) No feasible space.
[JNTU (IV B. Tech.) I Sem. Feb. 2007]

5. Define iso-profit and iso-cost lines. How do these help to obtain a solution to an LP problem?
[JNTU (IV B. Tech.) I Sem. June 2010]

2. Solution of Properly Behaved Problems

Example 1 *Find a geometrical interpretation and solution as well for the following LP problem :*

Maximize $z = 3x_1 + 5x_2$ *subject to restrictions :*

$x_1 + 2x_2 \leq 2000$, $x_1 + x_2 \leq 1500$, $x_2 \leq 600$, *and*

$x_1 \geq 0, x_2 \geq 0$.

Graphical Solution

Step 1. (To graph the inequality-constraints). Consider two mutually perpendicular lines OX_1 and OX_2 as axes of coordinates. Obviously, any point (x_1, x_2) in the positive quadrant will certainly satisfy non-negativity restrictions : $x_1 \geq 0$, $x_2 \geq 0$. To plot the line $x_1 + 2x_2 = 2000$, put $x_2 = 0$, find $x_1 = 2000$ from this equation.

Then mark a point L such that $OL = 2000$ by assuming a suitable scale, say 500 units = 2 cm. Similarly, again put $x_1 = 0$ to find $x_2 = 1000$ and mark another point M such that $OM = 1000$.

Now join the points L and M. This line will represent the equation $x_1 + 2x_2 = 2000$ as shown in *Fig. 1.*

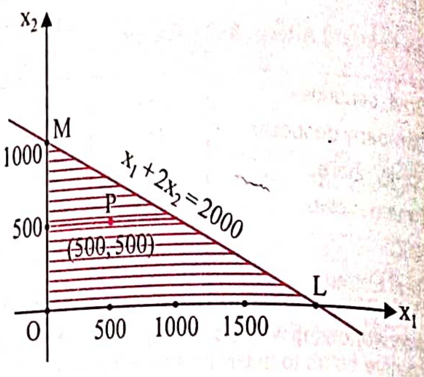


Fig. 1

Clearly, any point P lying on or below the line $x_1 + 2x_2 = 2000$ will satisfy the inequality $x_1 + 2x_2 \leq 2000$. (If we take a point (500, 500), *i.e.*, $x_1 = 500$, $x_2 = 500$, then we have $500 + 2 \times 500 < 2000$, which is true here).

Similar procedure is now adoped to plot the other two lines: $x_1 + x_2 = 1500$ and $x_2 = 600$ as shown in the *Fig. 2* and *Fig. 3*, respectively. Any point on or below the lines $x_1 + x_2 = 1500$ and $x_2 = 600$ will also satisfy other two inequalities : $x_1 + x_2 \leq 1500$, and $x_2 \leq 600$, respectively.

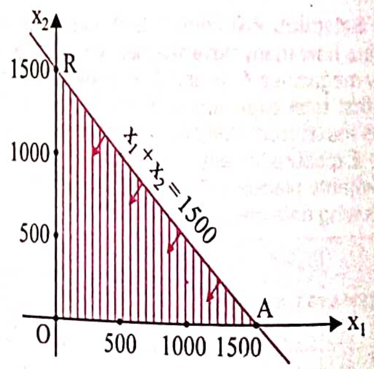


Fig. 2

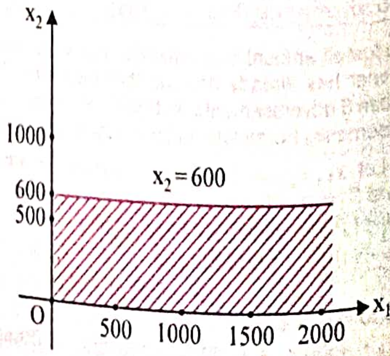


Fig. 3

Step 2. Find the *feasible region* or *solution space* by combining the *Figs.* 1, 2 and 3 together. A common shaded area

OABCD is obtained (see Fig. 4) which is a set of points satisfying the inequality constraints :

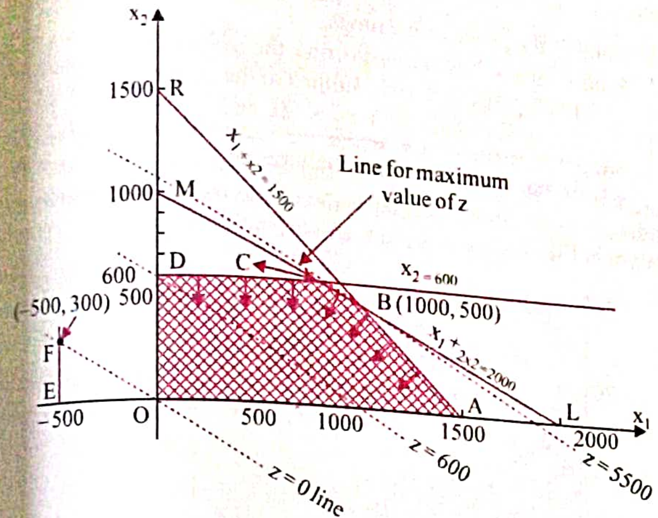


Fig. 4

$x_1 + 2x_2 \le 2000$, $x_1 + x_2 \le 1500$, $x_2 \le 600$,

and non-negativity restrictions as $x_1 \ge 0, x_2 \ge 0$. *Hence any point in the shaded area* (including its boundary) gives a feasible solution to the given LPP.

Step 3. Find the co-ordinates of the corner points of feasible region *O*, *A*, *B*, *C* and *D*.

Step 4. Locate the corner point of optimal solution either by calculating the value of **z** for each corner point *O*, *A*, *B*, *C*, and *D* (or by adopting the following procedure).

Here, the problem is to find the point or points in the feasible region (collection of all feasible solutions) which maximize(s) the objective (or profit) function. For some fixed value of **z**, $\mathbf{z} = 3x_1 + 5x_2$ is a straight line and *any point on it gives the same value of* **z**. Also, it should be noted that the lines corresponding to different values of **z** are parallel, because the gradient $(-3/5)$ of the line $\mathbf{z} = 3x_1 + 5x_2$ remains the same throughout. For $\mathbf{z} = 0$, *i.e.*, $0 = 3x_1 + 5x_2$, means a line which passes through the origin. To draw the line $3x_1 + 5x_2 = 0$, determine the ratio $\dfrac{x_1}{x_2} = \dfrac{-5}{3} = \dfrac{-500}{300}$.

Mark the point *E* moving 500 units distance from the origin on the negative side of X_1-axis. Then find the points *F* such that *EF* = 300 units in the positive direction of X_2-axis. Joining the point *F* and *O*, draw the line $3x_1 + 5x_2 = 0$. Now go on drawing the lines parallel to this line until at least a line is found which is farthest from the origin but passes through at least one corner of the feasible region at which the maximum value of **z** is attained. It is also possible that such a line may coincide with one of the edge of feasible region. In that case, every point on that edge gives the maximum value of **z**, thus having alternative solutions.

In this example, maximum value of **z** is attained at the corner point B(1000, 500), which is the point of intersection of lines $x_1 + 2x_2 = 2000$ and $x_1 + x_2 = 1500$. Hence, the required solution is $x_1 = 1000$, $x_2 = 500$ and max. value **z** = Rs. 5500.

NOTE If the number of vertices of feasible region is small, find the coordinates of vertices. As in above example, O = (0, 0), A = (1500, 0), B = (1000, 500), C = (800, 600), D = (0, 600) are obtained by solving the pair of lines whose intersections are these points, respectively. The value of *z* corresponding to these points will be $z_0 = 0$, $z_A = 4500$, $z_B = 5500$, $z_C = 4500$, $z_D = 3000$. Clearly $z_B = 5500$ is maximum for the point B (1000, 500) which gives the required solution.

Example 2 *Consider the problem :*
Max. $\mathbf{z} = x_1 + x_2$ *subject to the constraints,*

$$x_1 + 2x_2 \le 2000, x_1 + x_2 \le 1500, x_2 \le 600 \text{ and } x_1, x_2 \ge 0.$$

[IAS (Main) 2007 type]

Graphical Solution This problem is of the same type as discussed earlier except the objective function is slightly changed here. The feasible region will be similar to that of the above problem. Fig. 5 shows the objective function lines of the problem for three different values $\mathbf{z}_1$, $\mathbf{z}_2$, $\mathbf{z}_3$ of **z**.

It is clear from Fig. 5 that $\mathbf{z}_2$ is the maximum value of **z**. It is quite interesting that the line $\mathbf{z}_2$ representing the objective function lies along the edge *AB* of the polygon of feasible solutions. This indicates that the values of x_1 and x_2 which maximize **z** are not unique, but any point on the edge *AB* of *OABCD polygon will give the optimum value of* **z**. *The maximum value of* **z** *is always unique, but there will be an infinite number of feasible solutions which give unique value of* **z**. *Thus, two corners A* and *B as well as any other point on the line AB (segment) will give optimal solution of this problem.*

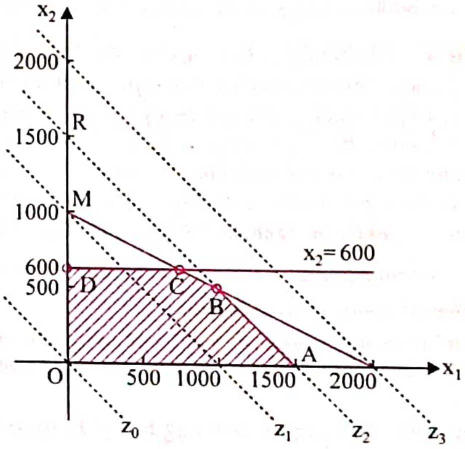


Fig. 5

It should be noted here that if a linear programming problem has more than one optimum solutions, there exist alternative optimum solutions. And, one of the optimum solutions will be corresponding to the corner point *B*, *i.e.* $x_1 = 1000$, $x_2 = 500$ with maximum profit **z** = Rs. 1500.

Example 3 *Solve the following LP problem graphically :*
Max. $z = 8000x_1 + 7000x_2$ *subject to he constraints :*

$$3x_1 + x_2 \le 66, \quad x_1 + x_2 \le 45, \quad x_1 \le 20, x_2 \le 40$$

and $x_1, x_2 \ge 0$.

Solution First, plot the lines $3x_1 + x_2 = 66$, $x_1 + x_2 = 45$, $x_1 = 20$ and $x_2 = 40$ and then shade the feasible region as shown in Fig. 6.

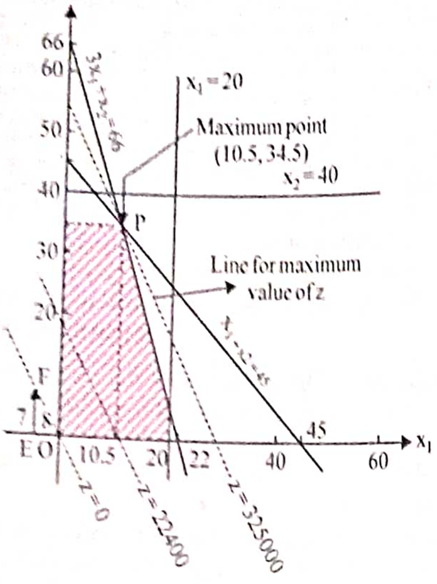


Fig. 6

Draw a dotted line $8000x_1+7000x_2=0$ for $z=0$ and continue to draw the lines till a point is obtained which is farthest from the origin but passing through at least one of the corners of the shaded (feasible) region. Fig. 6 shows that this point is $P(10{\cdot}5\,,\,34{\cdot}5)$ which is the point of intersection of lines

$$3x_1+x_2=66 \text{ and } x_1+x_2=45.$$

Hence, z is maximum for $x_1=10{\cdot}5$ and $x_2=34{\cdot}5$.

Max. $z=8000\times10{\cdot}5+7000\times34{\cdot}5=$ Rs. 325000.

Example 4 *Old hens can be bought at Rs. 2 each and young ones at Rs. 5 each. The old hens lay 3 eggs per week and the young ones lay 5 eggs per week, each egg being worth 30 paise. A hen (young or old) costs Re. 1 per week to feed. I have only Rs. 80 to spend for hens, how many of each kind should I buy to give a profit of more than Rs. 6 per week, assuming that I cannot house more than 20 hens.* **[JNTU (B. Tech. III, CS & Engg.) I.Sem. 2011, 2002]**

Solution *Formulation.* Let x_1 be the number of old hens and x_2 the number of young hens to be bought.

Since old hens lay 3 eggs per week and the yound ones lay 5 eggs per week, the total number of eggs obtained per week will be $=3x_1+5x_2$.

Consequently, the cost of each egg being 30 paise, the total gain will be= Rs. $0{\cdot}30(3x_1+5x_2)$.

Total expenditure for feeding (x_1+x_2) hens at the rate of Re. 1 each will be = Rs. $1.(x_1+x_2)$.

Thus, total profit z earned per week will be

$$\begin{aligned} z &= \text{Total gain} - \text{Total expenditure}\\ &= 0{\cdot}30(3x_1+5x_2)-(x_1+x_2)\\ &= 0{\cdot}50\,x_2-0{\cdot}10\,x_1 \textit{ (objective function)}. \end{aligned}$$

Since old hens can be bought at Rs. 2 each and young ones at Rs. 5 each and there are only Rs. 80 available for purchasing hens, the constraint is : $2x_1+5x_2\le 80$.

Also, since it is not possible to house more then 20 hens at a time, $x_1+x_2\le 20$.

Also, since the profit is restricted to be more than Rs. 6, this means that the profit function z is to be maximized. Thus there is no need to add one more constraint, *i.e.* $0{\cdot}5x_2-0{\cdot}1x_1\ge 6$.

Again, it is not possible to purchase negative quantity of hens, therefore $x_1\ge0\,,x_2\ge0$.

Finally, the problem becomes :
Find x_1 and x_2 so as to maximize the profit function,
$z=0{\cdot}50x_2-0{\cdot}10x_1$ subject to the constraints :
$2x_1+5x_2\le80,\ x_1+x_2\le20$, and $x_1\,,x_2\ge0$.

Graphical Solution. Plot the straight lines $2x_1+5x_2=80$ and $x_1+x_2=20$ on the graph and shade the feasible region as shown in Fig. 7.

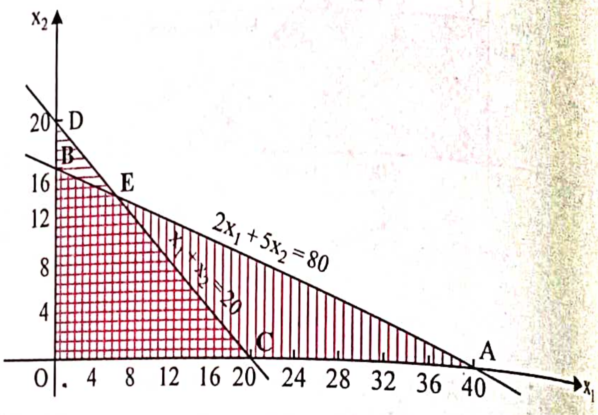


Fig. 7

The feasible region is *OBEC* . The coordinates of the extreme points of the feasible region are :

$O=(0\,,0)\,,\ C=(20\,,0)\,,\ B=(0\,,16)\,,$ and $E=(20/3\,,40/3)$.

The values of z at these vertices are :

$$z_o=0\,,\ z_C=0{\cdot}50\times0-0{\cdot}10\times20=-2\,,$$
$$z_B=0{\cdot}50\times16-0{\cdot}10\times0=8\,,$$
$$z_E=0{\cdot}50\times\frac{40}{3}-0{\cdot}10\times\frac{20}{3}=6\,.$$

Since the maximum value of z is Rs. 8 which occurs at the point $B=(0\,,16)$, the solution to the given problem is $x_1=0\,,x_2=16$, max. $z=$ Rs. 8.

Hence only 16 young hens I should buy in order to get the maximum profit of Rs. 8 (which is obviously > 6).

Example 5 (*Minimization problem*) *Consider the problem: Minimize* $z=1{\cdot}5x_1+2{\cdot}5x_2$ *subject to the constraints:*

$x_1+3x_2\ge3,\qquad x_1+x_2\ge2\,,\qquad x_1\,,x_2\ge0.$

Graphical Solution The geometrical interpretation of the problem is given in Fig. 8. The minimum value of z is

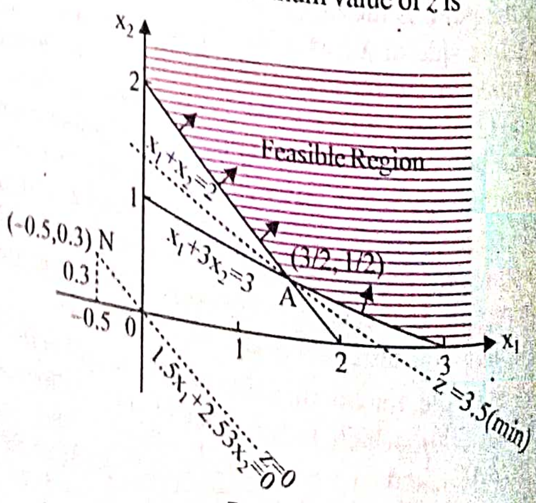


Fig. 8

$z_A = 3{\cdot}5$. This minimum is attained at the point of intersection A of the lines $x_1 + 3x_2 = 3$ and $x_1 + x_2 = 2$. This is the unique point to give the minimum value of z. Now, solving these two equations simultaneously, the optimum solution is :

$$x_1 = 3/2, x_2 = 1/2 \quad \text{and min. } z = 3{\cdot}5.$$

3. Graphical Solution in Some Exceptional Cases

The following examples show that there are some exceptional cases which must be taken into consideration if a general technique for solving LP problems is to be developed.

Example 6 ***(Problem having unbounded solution)***
Max $z = 3x_1 + 2x_2$ *subject to the constraints :*

$$x_1 - x_2 \le 1, \quad x_1 + x_2 \ge 3, \text{ and } x_1, x_2 \ge 0.$$

Graphical Solution The region of feasible solutions is the shaded area as shown in Fig. 9.

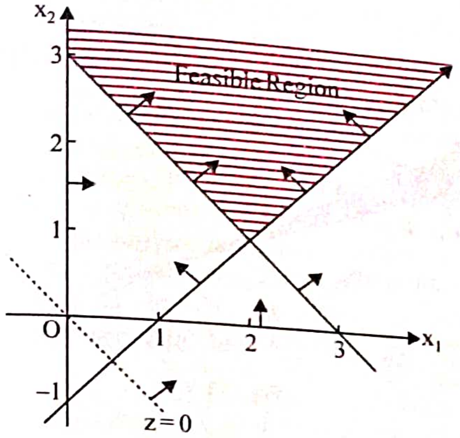


Fig. 9

It is clear from this figure that the line representing the objective function can be moved far even parallel to itself in the direction of increasing z, and still have some points in the region of feasible solutions.

Hence z can be made arbitrarily large, and the problem has no finite maximum value of z. Such problems are said to have *unbounded solutions.*

Infinite profit in practical problems of linear programming cannot be expected. If LP problem has been formulated by committing some mistake, it may lead to an unbounded solution.

Example 7 *Max.* $z = -3x_1 + 2x_2$ *subject to the constraints :*

$x_1 \le 3$, $x_1 - x_2 \le 0$, and $x_1, x_2 \ge 0$.

Graphical Solution In *Example* 6. it has been seen that both the variables can be made arbitrarily large as z is increased. In this example, an unbounded solution does not necessarily imply that all the variables can be made arbitrarily large as z approaches infinity. Here the variable x_1 remains constant as shown in Fig. 10.

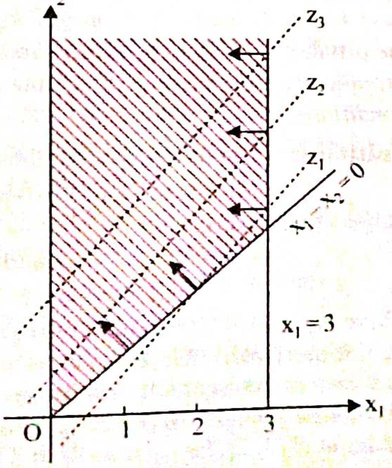


Fig. 10

Example 8 ***(Problem which is not completely normal)***
Maximize $z = -x_1 + 2x_2$ *subject to the constraints :*

$$-x_1 + x_2 \le 1, -x_1 + 2x_2 \le 4, \text{ and } x_1, x_2 \ge 0.$$

Graphical Solution The problem is solved graphically in Fig. 11.

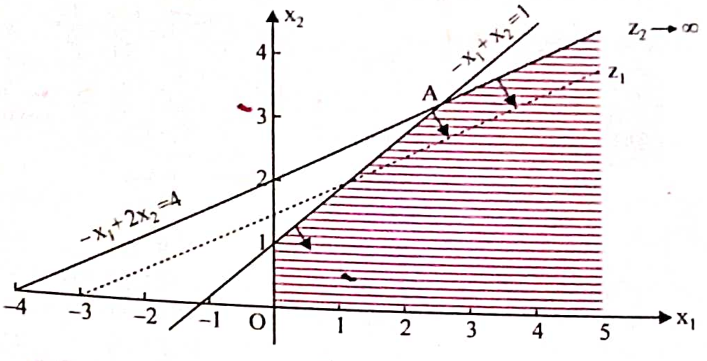


Fig. 11

In Fig. 11, the line of objective function coincides with the edge of Az_2 of the region of feasible solutions. Thus, every point (x_1, x_2) lying on this edge $(-x_1 + 2x_2 = 4)$, which is going to infinity on the right gives us $z = 4$, and is therefore an optimal solution.

Example 9 ***(Problem with inconsistent system of constraints)***
Maximize $z = 3x_1 - 2x_2$, *subject to the constraints :*

$$x_1 + x_2 \le 1 \qquad 2x_1 + 2x_2 \ge 4, \quad \text{and} \quad x_1, x_2 \ge 0.$$

Graphical Solution The problem is represented graphically in Fig. 12.

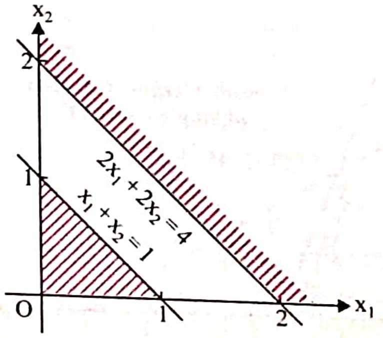


Fig. 12

The figure shows that there is no point (x_1, x_2) which satisfies both the constraints simultaneously. Hence the problem has no solution because the constraints are inconsistent.

Example 10 *(Constraints can be consistent and yet there may be no solution)*
Max. $z = x_1 + x_2$, *subject to* $x_1 - x_2 \ge 0$, $-3x_1 + x_2 \ge 3$, *and* $x_1, x_2 \ge 0$.

[IAS (Main) 2007 type]

Graphical Solution Fig. 13 shows that there is no region of feasible solutions in this case. Hence there is no feasible solution. So the question of having optimal solution does not arise.

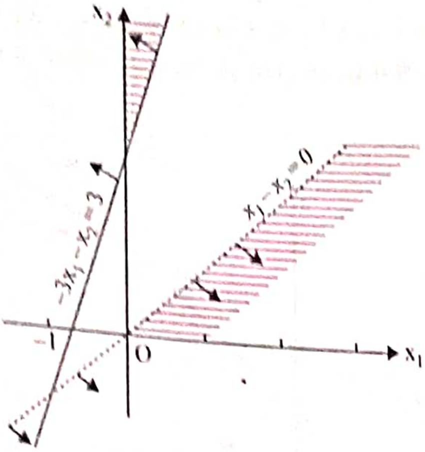


Fig. 13

Example 11 ***(Problem in which constraints are equations rather than inequalities)***
Maximize $z = 5x_1 + 3x_2$, *subject to the constraints :*

$$3x_1 + 5x_2 = 15, \quad 5x_1 + 2x_2 = 10, \quad x_1 \geq 0, x_2 \geq 0.$$

Graphical Solution Fig. 14 shows the graphical solution. Since there is only a single solution point $A\,(20/19\,,\,45/19)$, there is nothing to be maximized. Hence, a problem of this kind is of no importance. Such problems can arise only when the number of equations in the constratints is at least equal to the number of variables. If the solution is feasible, it is optimal. If it is not feasible, the problem has no solution.

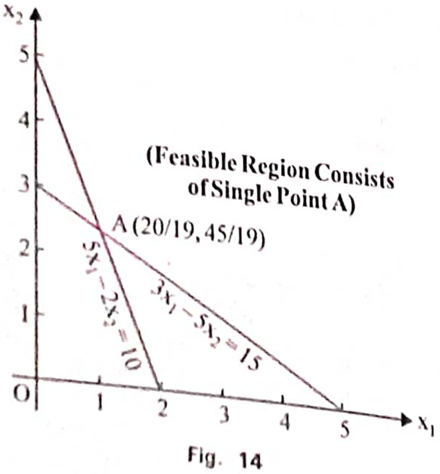


Fig. 14

Example 12 *A firm plans to purchase at least 200 quitals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintal of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantites. The percentage of X and Y metals in terms of weight in the scraps supplied by A and B is given below :*

Metals	*Supplier A*	*Supplier B*
X	25%	75%
Y	10%	20%.

The price of A's scrap is Rs. 200 per quintal and that of B's is R 400 per quintal. Formulate this problem as LP model and solve to determine the quantities that the firm should buy from the suppliers so as to minimize total purchase cost. **[Delhi (MBA) 199**

Solution The formulation of the given problem is :

$$\text{Min. (total cost) } Z = 200\,x_1 + 400\,x_2,$$

subject to the constraints :

$$x_1 + x_2 \geq 200, \frac{1}{4}x_1 + \frac{3}{4}x_2 \geq 100,$$

$$\frac{1}{10}x_1 + \frac{1}{5}x_2 \leq 35,$$

$$x_1 \geq 0,\ x_2 \geq 0,$$

where x_1, x_2 represent the number of quintals of scrap from two suppliers A and B, respectively.

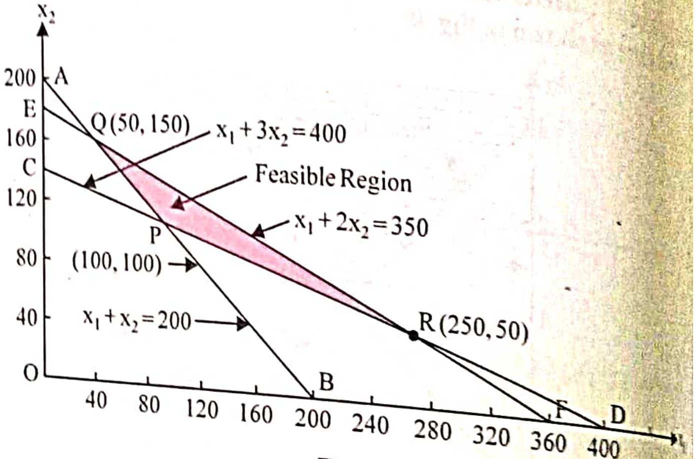


Fig. 15

In Fig. 15, the feasible region is the shaded area PQR which is obtained by drawing the graph of the constraints :

$$x_1 + x_2 \geq 200,\ x_2 + 3x_2 \geq 400 \text{ and } x_1 + 2x_2 \leq 175$$

The coordinates of the corner points of the feasible region are :

$$P\,(100, 100), \quad Q\,(50, 150), \quad R\,(250, 50).$$

The Z has the min. value at the point $P\,(100, 100)$. Thus the answer is $x_1 = 100, x_2 = 100$, min. $Z =$ Rs. 60,000.

Example 13 ***(Product Mix. Problem)*** *The standard weight of a special purpose brick is 5 kg and it contains two basic ingredients* B_1 *and* B_2. B_1 *costs Rs. 5 per kg and* B_2 *costs Rs. 8 per kg. Strength considerations state that the brick contains not more than 4 kg of* B_1 *and minimum of 2 kg of* B_2. *Since the demand for the product is likely to be related to the price of the brick, find out graphically minimum cost of the brick satisfying the above conditions.*

Solution The formulation of the given problem is :

$$\text{Minimize (total cost) } Z = 5x_1 + 8x_2$$

Subject to the constraints :

$$x_1 \leq 4, x_2 \geq 2 \text{ and } x_1 + x_2 = 5,$$

$$x_1 \geq 0,\ x_2 \geq 0,$$

where $(x_1, x_2) =$ the amount of ingredients B_1 (in kg) and B_2 (in kg.), respectively. The given constraints are plotted on the graph as shown in the figure. It may be observed that feasible region has two corner points $P(3, 2)$ and $Q(4, 2)$. The minimum value of Z is found at $P\,(3, 2)$, *i.e.* $x_1 = 3$, $x_2 = 2$. Hence the optimum product ix is to have 3 kg. of ingredient B_1 and 2 kg. of ingredient B_2 of a ecial case brick in order to achieve the minimum cost of Rs. 31.

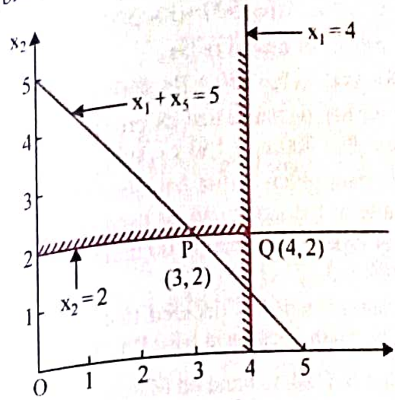


Fig. 16

MANAGEMENT APPLICATION

Example 14 *A local travel agent is planning a charter trip to a major sea resort. The eight-day seven-night package includes the fare for round-trip travel, surface transportation, board and lodging and selected tour options. The charter trip is restricted to 200 persons and past experience indicates that there will not be any problem for getting 200 persons. The problem for the travel agent is to determine the number of Deluxe, Standard, and Economy tour packages to offer for this charter. These three plans each differ according to seating and service for the flight, quality of accomodation, meal plans and tour options. The following table summarizes the estimated prices for the three packages and the corresponding expenses for the travel agent. The travel agent has hired an air craft for the flat fee of Rs. 2,00,000 for the entire trip.*

Price and costs for four packages per person

Tour plan	*Price (Rs.)*	*Hotel costs (Rs.)*	*Meals & Other expenses (Rs.)*
Deluxe	10,000	3,000	4,750
Standard	7,000	2,200	2,500
Economy	6,500	1,900	2,200

In planning the trip, the following considerations must be taken into account :

- (i) *At least 10 per cent of the packages must be of the deluxe type.*
- (ii) *At least 35 per cent but not more than 70 per cent must be of the standard type.*
- (iii) *At least 30 per cent must be of the economy type.*
- (iv) *The maximum number of deluxe packages available in any air craft is restricted to 60.*
- (v) *The hotel desires that at least 120 of the tourists should be on the deluxe and standard packages together.*

The travel agent wishes to determine the number of packages to offer in each type so as to maximize the total profit.

- (a) *Formulate the above as a linear programming problem.*
- (b) *Restate the above linear programming problem in terms of two decision variables, taking advantage of the fact that 200 packages will be sold.*
- (c) *Find the optimum solution using graphical method for the restated linear programming problem and interpret your results.* **[C.A. (May 91)]**

Solution Let x_1, x_2, x_3 be the number of Deluxe, Standard & Economy tour packages restricted to 200 persons only to maximize the profits of the concern.

The contribution (per person) arising out of each type of tour package offered is as follows :

Package type offered	Price (Rs.)	Hotel Costs (Rs.)	Meals, etc. (Rs.)	Net profit (Rs.)
	(1)	(2)	(3)	(4) = (1) – [(2) + (3)]
Deluxe	10,000	3,000	4,750	2,250
Standard	7,000	2,200	2,500	2,300
Economy	6,500	1,900	2,200	2,400

Since the travel agent has to pay the flat fee of Rs. 2,00,000 for the chartered aircraft for the entire trip, the profit function will be :

Max. $P = \text{Rs.}(2250x_1 + 2300x_2 + 2400x_3) - \text{Rs. } 2,00,000.$

The constraints according to the given conditions *(i)* to *(v)* are as follows :

$x_1 \geq 20$ from *(i)* $\quad x_3 \geq 60$ from *(iii)* $\quad x_1 + x_2 + x_3 = 200,$

$x_2 \geq 70$ from *(ii)* $\quad x_1 \leq 60$ from *(iv)* $\quad x_2 \leq 140$ from *(ii)*

$x_1 + x_2 \geq 120$ from *(v)* and $x_1, x_2, x_3 \geq 0.$

In compact form, above constraints can be reduced to the following forms :

$$20 \leq x_1 \leq 60,\ 70 \leq x_2 \leq 140,\ x_3 \geq 60, x_1 + x_2 \geq 120,$$

$$x_1 + x_2 + x_3 = 200 \text{ and } x_1, x_2, x_3 \geq 0.$$

(a) The linear programming formation is as given above.

(b) Since $x_1 + x_1 + x_3 = 200$, *i.e.* $x_3 = 200 - (x_1 + x_2)$, substitute the value of x_3 in the above relations to get the following reduced LPP :

$$Max.\ P = -150x_1 - 100x_2 + 2,80,000 \text{ subject to}$$

$$20 \leq x_1 \leq 60,\ 70 \leq x_2 \leq 140,\ 120 \leq x_1 + x_2 \leq 140$$

and $$x_1, x_2 \geq 0.$$

(c) **Graphical Solution.** Refer to the following figure for the restated LP. problem as in *(b)*.

From above figure, we compute

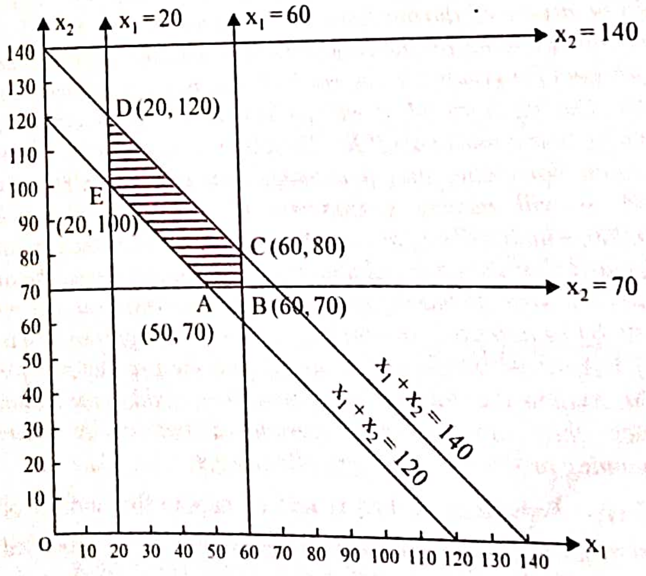


Fig. 17.

Corner points	Coordinates of corner points	Values of objective function : $P = -150x_1 - 100x_2 + 2,80,000$
A	(50, 70)	P_A = Rs. 2,65,500
B	(60, 70)	P_B = Rs. 2,64,000
C	(60, 80)	P_C = Rs. 2,63,000
D	(20, 120)	P_D = Rs. 2,65,000
E	(20, 100)	P_E = Rs. 2,67,000

Thus maximum profit is attained at the corner point (20, 100).

Interpretation of Solution. Maximum profit of Rs. 2,67,000 is attained when $x_1 = 20$, $x_2 = 100$ and $x_3 = 200 - (x_1 + x_2) = 80$.

In other words, the travel agent should offer 20 *delux*, 100 *standard* and 80 *economy* tour packages so as to get the maximum profit of Rs. 2,67,000.

Example 15 *(Product Mix Problem) Semicond is an electronics company manufacturing tape recorders and radios. Its per unit labour costs, raw material costs and selling prices are given in Table 1. An extract from its balance sheet on 31.3.1994 is shown in Table 2. Its current asset/current liability ratio (called the current ratio) is 2.*

Table 1 : *Cost Information*

For Products	Selling Price	Labour Cost	Raw Material Cost
Tape Recorder	Rs. 1,000	Rs. 500	Rs. 300
Radio	Rs. 900	Rs. 350	Rs. 400

Table 2 : *Extract from Balance Sheet as on 31.3.1994*

Current Liabilities (Rs.)	Current Assets (Rs.)
Cash	1,00,000
* Accounts Receivable	30,000
** Inventory	70,000
Short-Term Bank Borrowing	1,00,000

* *Accounts receivable is amount due from customers.*

** *100 units of raw material used for tape recorder and 100 units of raw material used for radio.*

Semicond must determine how many tape recorders and radios should be produced during April 94. Demand is large enough to ensure that all goods produced will be sold. All sales are on credit and payment for goods sold in April 94 will not be received until 31.5.94. During April 94, it will collect Rs. 20,000 in accounts receivable and it must payoff Rs. 10,000 of the outstanding short term bank borrowing and a monthly rent of Rs. 10,000. On 30.4.94, it will receive a shipment of raw material worth Rs. 20,000, which will be paid on May 31, 1994. The management has decided that the cash balance on April 30, 1994 must be at least 40,000. Also, its banker requires that the current ratio as on April 30, 94 be at least 2. In order to maximize the contribution to profit for April 94 production it has to find the product mix for April 94. Assume that labour costs (wages) are paid in the month in which they are incurred. Formulate this as a linear programming problem and graphically solve it.

Solution *Formulation.* Let x_1 and x_2 denote the number of units of tape recorders and radios respectively to be produced during April 1994.

Profit per unit of tape recorder

= Selling price − (Labour cost + Raw material cost)

= Rs. 1000 − (Rs. 500 + Rs. 300) = Rs. 200.

Similarly, profit per unit of radio

= Rs. 900 − (Rs. 350 + Rs. 400) = Rs. 150.

Company wishes to maximize its profit, therefore objective function is : Max. $P = 200x_1 + 150x_2$, subject to the constraints :

(1) As per data given in the balanced sheet, the inventory available in the stock can be used only to produce 100 units of tape recorder and 100 units of radio. Therefore, $x_1 \le 100$ and $x_2 \le 100$.

(2) The management has decided that the cash balance on April 30, 1994 must be at least Rs. 40,000.

Cash balance = Cash in hand on March 31, 94

+ Accounts receivable collected in April 94

− Bank borrowing paidoff in April 94

− Monthly rent paid − Labour cost paid during April 94.

= Rs. 1,00,000 + Rs. 20,000 − Rs. 10,000

− Rs. 10,000 − $(500x_1 + 350x_2)$

Management wants cash balance ≥ Rs. 40,000

or $\quad$ Rs. $1,00,000 - 500x_1 - 350x_2 \ge 40,000$ ...(*i*)

or $\quad$ Rs. $60,000 \ge 500x_1 + 350x_2$

or $\quad$ $500x_1 + 350x_2 \le$ Rs. 60,000. ...(*ii*)

(3) Bankers require current ratio as on (April 30, 1994) ≥ 2,

Current ratio = Current assets/*Current liabilities.*

Now we have to find the value of cash balance, accounts receivable, inventory and current liabilities as on April 30, 1994.

Cash balance = Rs. $1,00,000 - 500x_1 - 350x_2$ ...[from (*i*)]

Accounts receivables as on April 30, 1994

= Accounts receivable on March 31, 1994

+ Accounts receivable due from April sale

− Accounts receivable collected during April

= Rs. 30,000 + $(1000x_1 + 900x_2)$ − Rs. 20,000

= Rs. $10,000 + 1000x_1 + 900x_2$.

Inventory on April 30, 1994

= Inventory as on March 31, 1994

+ Inventory received during April, 1994

− Inventory consumed during April, 1994

= Rs. 70,000 + Rs. 20,000 − $(300x_1 + 400x_2)$

= Rs. 90,000 − $(300x_1 + 400x_2)$.

Current assets as on April 30, 1994

= Cash balance + Accounts receivables

Inventory value on April 30, 1994

= Rs. $1,00,000 - 500x_1 - 350x_2$

+ Rs. $10,000 + 1000x_1 + 900x_2$

+ Rs. $90,000 - 300x_1 - 400x_2$

= Rs. $2,00,000 + 200x_1 + 150x_2$.

Current liabilities as on April 30, 1994

= Value of bank borrowings as on March, 94

− Loan paid during April, 1994

+ Amount due on inventory received during April 1994

= Rs. 1,00,000 − Rs. 10,000 + Rs. 20,000 = Rs. 1,10,000.

But, bank requires that current ratio as on April 30, 1994 be at ast 2

That is, current assets/current liabilities are ≥ 2,

$$\left(\frac{\text{Rs. } 2{,}00{,}000 + 200\,x_1 + 150\,x_2}{\text{Rs. } 1{,}10{,}000}\right) \geq 2$$

$$\text{Rs. } 2{,}00{,}000 + 200\,x_1 + 150\,x_2 \geq \text{Rs. } 2{,}20{,}000$$

$$200\,x_1 + 150\,x_2 \geq \text{Rs. } 20{,}000. \qquad ...(iii)$$

Thus, the linear programming model for the Semicond is as llows :

Maximize $\mathbf{P} = 200\,x_1 + 150\,x_2$, subject to the constraints :

$x_1 \leq 100, x_2 \leq 100, 500\,x_1 + 350\,x_2 \leq 60000,$
$200\,x_1 + 150\,x_2 \geq 20000$ and $x_1 \geq 0, x_2 \geq 0.$

Graphical Solution The feasible region enclosed by the nstraints is given by points *A, B, C, D* with their coordinates :

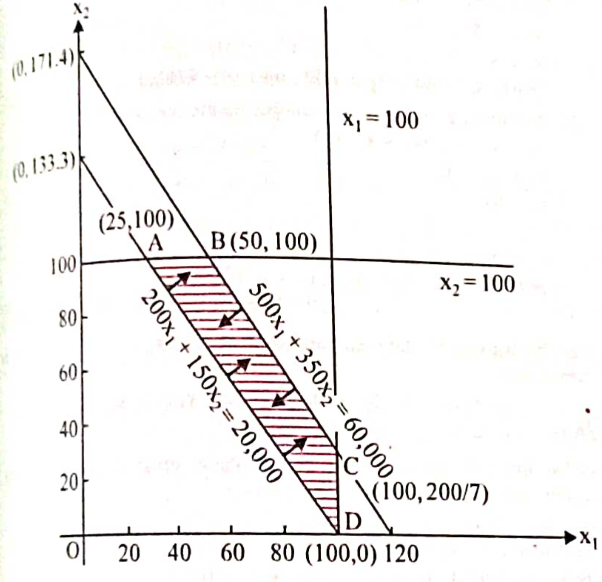


Fig. 18

$A(25, 100), B(50, 100), C(100, \frac{200}{7}), D(100,0).$

The profit at these coordinates is found as below :

$A(25, 100)$: Rs. 200 × 25 + Rs. 150 × 100 = Rs. 20,000

$B(50, 100)$: Rs. 200 × 50 + Rs. 150 × 100 = Rs. 25,000

$C(100, 200/7)$: Rs. 200 × 100 + Rs. 150 × 200 / 7 = Rs. 24285·7

$D(100, 0)$: Rs. 200 × 100 + Rs. 150 × 0 = Rs. 20,000.

Since maximum profit is attained at the point $B(50, 100)$, Semicond can maximize its profit by producing 50 tap recorders and 100 radios during April, 1994 and the total profit contribution will be Rs. 25,000.

Q. Explain (*i*) No feasible solution, (*ii*) Unbounded solution. Give one example in each case.

4. Important Geometric Properties of LP Problems

Geometric properties of LP problems, observed while solving them graphically, are summarized as below :

1. The region of feasible solutions has an important property which is called the *convexity property* in geometry, provided the feasible solution of the problem exists.

   Convexity means that region of feasible solutions has *no holes* in them, that is, they are solids, and they have no cuts (like /\/\/\/\/\/\/\/) on the boundary. This fact can be expressed more precisely by saying that the line joining any two points in the region also lies in the region.
2. The boundaries of the regions are lines or planes.
3. There are corners or extreme points on the boundary, and there are edges joining various corners.
4. The objective function can be represented by a line or a plane for any fixed value of z.
5. At least one corner of the region of feasible solutions will be an optimal solution whenever the maximum or minimum value of z is finite.
6. If the optimal solution is not unique, there are points other than corners that are optimal but in any case at least one corner is optimal.
7. The different situation is found when the objective function can be made arbitrarily large. Of course, no corner is optimal in that case.

Examination PROBLEMS

1. Solve the following LP problems by graphical method :

(a) Min $z = 5x_1 - 2x_2$; s.t. $2x_1 + 3x_2 \geq 1, x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are : $(\frac{1}{2}, 0), (0, \frac{1}{3})$]
[Ans. $x_1 = 0, x_2 = 1/3$, min. $z = -2/3$.]

(b) Max $z = 5x_1 + 3x_2$; s.t. $3x_1 + 5x_2 \leq 15, 5x_1 + 2x_2 \leq 10$; $x_1, x_2 \geq 0$ **[IAS (main) 2011]**
[Hint. Vertices of the feasible region are : (0, 0), (2, 0), (20 / 19, 45 / 19) and (0, 3)]
[Ans. $x_1 = 20/19, x_2 = 45/19$, max. $z = 235/19$]

(c) Max $z = 2x_1 + 3x_2$; s.t. $x_1 + x_2 \leq 1, 3x_1 + x_2 \leq 4 : x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are :
(0, 0), (1, 0), (0, 1)]
[Ans. $x_1 = 0, x_2 = 1$, max. $z = 3$]

(d) Max $z = 5x_1 + 7x_2$; s.t. $x_1 + x_2 \leq 4, 3x_1 + 8x_2 \leq 24$, $10x_1 + 7x_2 \leq 35, x_1, x_2 \geq 0$.
[JNTU (IV B. Tech.) I Sem. 2006; Meerut 90]
[Hint. Vertices of the feasible region are :
(0, 0), (7 / 2, 0), (7 / 3, 5 / 3), (8 / 5, 12 / 5) and (0, 3)]
[Ans. $x_1 = 8/5, x_2 = 12/5$, max. $z = 124/5$]

(e) Min $z = -x_1 + 2x_2$; s.t. $-x_1 + 3x_2 \leq 10, x_1 + x_2 \leq 6$, $x_1 - x_2 \leq 2, x_1, x_2 \geq 0$.
[Hint. Vertices of the feasible region are :
(0, 0) (2, 0), (4, 2), (2, 4) and (0, 10 / 3)]
[Ans. $x_1 = 2, x_2 = 0$, min. $z = -2$]

(f) Min $z = 20x_1 + 10x_2$; s.t. $x_1 + 2x_2 \leq 40, 3x_1 + x_2 \geq 30$, $4x_1 + 3x_2 \geq 60$, and $x_1 \geq 0, x_2 \geq 0$.
[JNTU (IV B. Tech, ME etc.) I Sem., May 2011 Feb. 2007]
[Hint. Vertices of the feasible region are :
(15, 0) (40, 0), (4, 18) and (6, 12)]
[Ans. $x_1 = 6, x_2 = 12$, min. $z = 240$]

(g) Maximize $z = 3x_1 + 4x_2$ subject to :

$$x_1 - x_2 \leq -1 - x_1 + x_2 \leq 30 ; x_1, x_2 \geq 0.$$

[JNTU (IV B. Tech. Mech,) I Sem. 2011; Bhub. (IT) 2004]
[Ans. The problem has no solution.]

Applications of Simplex Method

BASIC CONCEPTS

Here the theoretical results obtained earlier be used for computational development of simplex method. In this method, we approach from initial basic feasible solution (extreme point) to new one (having a value of z, at least as large as the preceding one) until an optimal solution is reached.

The details of simplex method are developed for solving standard LPP : Max $z = \mathbf{cx}$, subject to $A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0$ where A is a matrix. For convenience, we shhall take maximization problem only.

It has not been possible to obtain the graphical solution to the LP problem of more than two variables. The analytic solution is also not possible because the tools of analysis are not well suited to handle inequalities. In such cases, a simple and most widely used simplex method is adopted which was developed by ***G. Dantzig*** in 1947.

The *simplex method* provides an ***algorithm*** (a rule of procedure usually involving repetitive application of a prescribed operation) which is based on the ***fundamental theorem of linear programming.***

It is clear from Fig. 4 (page LPP/17) that feasible solutions may be *infinite* in number (because there are infinite number of points in the feasible region, *OABCD*). So, it is rather impossible to search for the optimum solution amongst all the feasible solutions. But fortunately, the number of basic feasible solutions are finite in number (which are corresponding to extreme points O, A, B, C, D, respectively). Even then, a great labour is required in finding all the basic feasible solutions and to select that one which optimizes the objective function.

The simplex method provides a systematic algorithm which consists of moving from one basic feasible solution (one vertex) to another in a prescribed manner so that the value of the objective function is improved. This procedure of jumping from vertex to vertex is repeated. If the objective function is improved at each jump, then no basis can ever repeat and there is no need to go back to vertex already covered. Since the number of vertices is finite, the process must lead to the optimal vertex in a finite number of steps. The procedure is explained in detail through a numerical example (*see Example 1,* on page SMX/17).

The simplex algorithm is an iterative (*i.e.* step-by-step) procedure for solving LP problems. It consists of—

(i) having a trial basic feasible solution to constraint-equations,

(ii) testing whether it is an optimal solution, or not

(iii) improving the first trial solution by a set of rules, and repeating the process till an optimal solution is obtained.

The computational procedure requires at most m [equal to the number of equations in (2),] non-zero variables in the solution at any step. In case of less than m non-zero variables at any stage of computations the degeneracy arises in LP problem. The case of degeneracy has also been discussed in detail in *this chapter*.

Further, it is very interesting to note that a feasible solution at any iteration is related to the feasible solution at the successive iteration in the following way. One of the non-basic variables (which are zero now) at one iteration becomes *basic* (non-zero) at the following iteration, and is called an ***entering variable.*** To compensate, one of the basic variables (which are non-zero now) at one iteration becomes non-basic (zero) at the following iteration, and is called a ***departing variable.*** The other non-basic variables remain zero, and the other basic variables, in general, remain non-zero (though their values may change).

For convenience, we re-state the LP problem in standard form :

$$\text{Max. } z = c_1x_1 + c_2x_2 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m} \quad \ldots(1)$$

subject to the constraints :

$$\left.\begin{array}{llll} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n & + x_{n+1} & & = b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots. + a_{2n}x_n & & + x_{n+2} & = b_2 \\ \ldots \quad \ldots \quad \ldots & \ldots & \ldots & \ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots. + a_{mn}x_n & & + x_{n+m} & = b_m \end{array}\right\} \quad \ldots(2)$$

and

$$x_1 \geq 0, x_2 \geq 0, \ldots., x_n \geq 0, x_{n+1} \geq 0, \ldots., x_{n+m} \geq 0 \quad \ldots(3)$$

For easiness, an obvious starting basic feasible solution of m equations (2) is usually taken as : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; $x_{n+1} = b_1$, $x_{n+2} = b_2, \ldots, x_{n+m} = b_m$. For this solution, the value of the objective function (1) is zero. Here $x_1, x_2, x_3, \ldots, x_n$ (each equal to zero) are ***non-basic variables*** and remaining variables $(x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m})$ are ***basic variables*** (some of them may also have the value zero).

SUMMARY OF DEFINITIONS AND NOTATIONS

The first basic feasible solution is : $x_1 = x_2 = x_3 = \ldots = x_n = 0$; and $x_{n+1} = b_1$, $x_{n+2} = b_2$, $x_{n+3} = b_3 \ldots$, $x_{n+m} = b_m$ for the reformulated LP problem : Max $\mathbf{z} = \mathbf{CX}$, subject to $\mathbf{AX} = \mathbf{b}$ and $\mathbf{X} \geq \mathbf{0}$.

First denote the jth column of $m \times (n+m)$ matrix **A** by $\mathbf{a}_j$ $(j = 1, 2, 3, \ldots, n+m)$, so that

$$\mathbf{A} = [\mathbf{a_1}, \mathbf{a_2}, \ldots, \mathbf{a_{m+n}}]. \quad \ldots(1)$$

Now form an $m \times m$ non-singular matrix **B**, called ***basis matrix,*** whose column vectors are m linearly independent columns selected from matrix **A** and renamed as $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$ .Therefore,

$$\mathbf{B} = [\beta_1, \beta_2, \ldots, \beta_m] = [\mathbf{a_{n+1}}, \mathbf{a_{n+2}}, \ldots, \mathbf{a_{n+m}}] \quad \ldots(2)$$

For initial basic feasible solution,

$$\mathbf{B} = [(\mathbf{1}, \mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}), (\mathbf{0}, \mathbf{1}, \mathbf{0}, \mathbf{0} \ldots, \mathbf{0}), \ldots, (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{1})] = \boldsymbol{I_m} \textit{ (identity matrix)}.$$

The matrix **B** is evidently a basis matrix because column vectors in **B** form a basis set of m-dimensional Euclidean space (E^m).

Second, denote the basic variables $x_{n+1}, x_{n+2}, \ldots, x_{n+m}$ by $x_{B1}, x_{B2}, \ldots, x_{Bm}$ respectively, to give the basic feasible solution in the form :

$$\mathbf{X}_B = (x_{B1}, x_{B2}, x_{B3}, \ldots, x_{Bm}) = (x_{n+1}, x_{n+2}, x_{n+3}, \ldots, x_{n+m}) \quad \ldots(3)$$

For initial basic feasible solution,

$$\mathbf{X}_B = (b_1, b_2, b_3, \ldots, b_m) = \text{ right side constants of (2).}$$

Next, the coefficients of basic variables $x_{B1}, x_{B2}, \ldots, x_{Bm}$ in the objective function z will be denoted by $c_{B1}, c_{B2}, \ldots, c_{Bm}$, respectively, so that

$$\mathbf{C_B} = (\boldsymbol{c_{B1}}, \boldsymbol{c_{B2}}, \ldots, \boldsymbol{c_{Bm}}).$$

For initial basic feasible solution,

$$\mathbf{C}_B = (\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}) = \mathbf{0} \textit{ (null vector)}$$

Consequently, the objective function $z = c_1x_1 + c_2x_2 + c_3x_3 + \ldots + c_nx_n + 0x_{n+1} + 0x_{n+2} + \ldots + 0x_{n+m}$ becomes

$$z = c_1.0 + c_2.0 + \ldots + c_n.0 + c_{B1}x_{B1} + \ldots + c_{Bm}x_{Bm} \quad [\text{since } x_1 = x_2 = x_3 = \ldots = x_n = 0]$$

or

$$z = \mathbf{C}_B \, \mathbf{X}_B . \quad \ldots(4)$$

Because $\mathbf{C}_B = \mathbf{O}$ (null vector) for initial solution, therefore

$$z = \mathbf{0}, \mathbf{X_B} = \mathbf{b}.$$

Since **B** is an $m \times m$ non-singular basis matrix, any vector in E^m can be expressed as a linear combination of vectors in **B** (by definition of basis for vector space). In particular, each vector $\mathbf{a}_j$ $(j = 1, 2, \ldots, n+m)$ of matrix **A** can be expressed as a linear combination of vectors β_i $(i = 1, 2, .., m)$ in **B**. The notation for such linear combination is given by

$$\mathbf{a}_j = x_{1j} \, \beta_1 + x_{2j} \, \beta_2 + \ldots + x_{mj} \, \beta_m = (\beta_1, \beta_2, \ldots, \beta_m) \begin{bmatrix} x_{1j} \\ : \\ x_{mj} \end{bmatrix} = \mathbf{BX}_j \quad \ldots(5)$$

where x_{ij} $(i = 1, 2, 3, .., m)$ are scalars required to express each $\mathbf{a}_j$ $(j = 1, 2, 3, \ldots, n+m)$ as linear combination of basis vectors $\beta_1, \beta_2, \beta_3, \ldots, \beta_m$.

Therefore, $\mathbf{X}_j = \mathbf{B}^{-1}\mathbf{a}_j$ and hence matrix $(\mathbf{X}_j)$ will change if the columns of (**A**) forming (**B**) change.

For initial solution, $\mathbf{a}_j = \mathbf{I}_m\mathbf{X}_j = \mathbf{X}_j$.

Next define a new variable, say z_j , as

$$z_j = x_{1j} \, c_{B1} + x_{2j} \, c_{B2} + \ldots + x_{mj} c_{Bm} = \sum_{i=1}^{m} c_{Bi} x_{ij} = \mathbf{C}_B \mathbf{X}_j . \quad \ldots(6)$$

Δ_j denotes the *net evaluation* which is computed by the formula :

$$\Delta_j = z_j - c_j = C_B X_j - c_j \qquad ...(7)$$

Lastly, these notations can be summerized in the following *starting simplex table.*

Table 1 : Starting Simplex Table

		$c_j \rightarrow$	c_1	c_2	...	c_n	0	0	...	0	
BASIC VARIABLES	$\mathbf{C}_B$	$\mathbf{X}_B$	$\mathbf{X}_1 (= \mathbf{a}_1)$	$\mathbf{X}_2 (= \mathbf{a}_2)$	...	$\mathbf{X}_n (= \mathbf{a}_n)$	$\mathbf{X}_{n+1}$ (β_1)	$\mathbf{X}_{n+2}$ (β_2)	...	$\mathbf{X}_{n+m}$ (β_m)	**MINIMUM RATIO**
$x_{n+1} (= s_1)$	$c_{B1} (= 0)$	$x_{B1} (= b_1)$	$x_{11} (= a_{11})$	$x_{12} (= a_{12})$	...	$x_{1n} (= a_{1n})$	1	0	...	0	
$x_{n+2} (= s_2)$	$c_{B2} (= 0)$	$x_{B2} (= b_2)$	$x_{21} (= a_{21})$	$x_{22} (= a_{22})$	...	$x_{2n} (= a_{2n})$	0	1	...	0	
:	:	:	:	:	:	:	:	:	:	:	
$x_{n+m} (= s_m)$	$c_{Bm} (= 0)$	$x_{Bm} (= b_m)$	$x_{m1} (= a_{m1})$	$x_{m2} (= a_{m2})$	...	$x_{mn} (= a_{mn})$	0	0	...	1	
	$z = \mathbf{C}_B \mathbf{X}_B$		Δ_1	Δ_2	...	Δ_n	0	0	...	0	$\leftarrow \Delta_j = \mathbf{C}_B \mathbf{X}_j - c_j$

NOTE Basic variables in the first column are always sequenced in the order of columns forming the unit matrix in the table.

Above definitions and notations can be clearly understood by the following numerical example.

Example : *Illustrate definitions and notations by the linear programming problem :*
Maximize $z = x_1 + 2x_2 + 3x_3 + 0x_4 + 0x_5$, *subject to* $4x_1 + 2x_2 + x_3 + x_4 = 4$, $x_1 + 2x_2 + 3x_3 - x_5 = 8$.

Solution First of all, constraint equations in matrix form may be written as

$$\overset{\mathbf{A}}{\begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ \hline 4 & 2 & 1 & 1 & 0 \\ 1 & 2 & 3 & 0 & -1 \end{array}} \overset{\mathbf{X}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}} = \overset{\mathbf{b}}{\begin{bmatrix} 4 \\ 8 \end{bmatrix}}$$

or

$$\mathbf{AX} = \mathbf{b}.$$

A basis matrix $\mathbf{B} = (\beta_1, \beta_2)$ is formed using columns $\mathbf{a}_3$ and $\mathbf{a}_1$, so that

$$\beta_1 = \mathbf{a}_3 = \begin{pmatrix} 1 \\ 3 \end{pmatrix}, \beta_2 = \mathbf{a}_1 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

The rank of matrix **A** is 2, and hence $\mathbf{a}_3$, $\mathbf{a}_1$ column vectors are linearly independent, and thus forms a basis for R^2.

Thus, basis matrix is

$$\mathbf{B} = (\beta_1, \beta_2) = \begin{array}{cc} \mathbf{a}_3 & \mathbf{a}_4 \\ \end{array} \begin{pmatrix} 1 & 4 \\ 3 & 1 \end{pmatrix}$$

Using (4) *on p. SMX/1*, the basic feasible solution is

$$\mathbf{X_B} = \mathbf{B}^{-1} \mathbf{b} = \left[\frac{1}{|\mathbf{B}|} \text{adj}(\mathbf{B}) \right] \mathbf{b} = \frac{-1}{11} \begin{bmatrix} 1 & -4 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 8 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 28 \\ 4 \end{bmatrix}$$

or

$$\mathbf{X_B} = \begin{bmatrix} 28/11 \\ 4/11 \end{bmatrix} = \begin{bmatrix} x_{B1} \\ x_{B2} \end{bmatrix}.$$

Therefore, basic variables are $x_{B1} = 28/11 = x_3$, $x_{B2} = 4/11 = x_1$, and remaining variables are non-basic (which are always zero), *i.e.* $x_2 = x_4 = x_5 = 0$. Also,

$$c_{B1} = \text{coefficient of } x_{B1} = \text{coeff. of } x_3 = c_3 = 3$$
$$c_{B2} = \text{coefficient of } x_{B2} = \text{coeff. of } x_1 = c_1 = 1.$$

Hence $\qquad c_B = (3, 1).$

Now, using (5) *on p. SMX/2*, the value of the objective function is

$$z = \mathbf{C}_B \mathbf{X}_B = (3, 1) \begin{pmatrix} 28/11 \\ 4/11 \end{pmatrix} = \frac{88}{11}.$$

Also, any vector $\mathbf{a}_j = (j = 1, 2, 3, 4, 5)$ can be expressed as linear combination of vectors β_i $(i = 1, 2)$. Therefore, to express $\mathbf{a}_2$ as linear combination of β_1, β_2, we have

$$\mathbf{a}_2 = x_{12}\, \beta_1 + x_{22}\, \beta_2 = x_{12}\, \mathbf{a}_3 + x_{22}\, \mathbf{a}_1.$$

To compute values of scalars x_{12} and x_{22}, use the result (3) on p. SMX/1 to get

$$\mathbf{X}_2 = \mathbf{B}^{-1} \mathbf{a}_2 = -\frac{1}{11} \begin{pmatrix} 1 & -4 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \begin{pmatrix} x_{12} \\ x_{22} \end{pmatrix}$$

Therefore $x_{12} = 6/11, x_{22} = 4/11$.

Similar treatment can be adopted for expressing other $\mathbf{a}_j$'s as linear combinations of β_1 and β_2.

Now, using (6b) *on page SMX/2*, the variable z_2 corresponding to vector $\mathbf{a}_2$ can be obtained as

$$z_2 = \mathbf{C}_B \mathbf{X}_2 = (3, 1)\begin{pmatrix} 6/11 \\ 4/11 \end{pmatrix} = \left(3 \times \frac{6}{11} + 1 \times \frac{4}{11}\right) = \frac{22}{11}.$$

Similarly, z_1, z_3, z_4, z_5 can also be computed.

COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

The computational aspect of simplex method can be easily understood by the following simple example.

Example 1 *Consider the linear programming problem Maximize* $z = 3x_1 + 2x_2$ *subject to the constraints :*

$$x_1 + x_2 \le 4, x_1 - x_2 \le 2, \textit{and } x_1, x_2 \ge 0.$$

[JNTU (MBA) II Sem. 2010; IAS (Maths.) 2007 (Type), 2004; Kanpur 2000]

Solution *Step 1.* First, observe whether all the right side constants of the constraints are non-negative. If not, it can be changed into positive value on multiplying both sides of the constraints by – 1. In this example, all the b_i' s (right side constants) are already positive.

Step 2. Next convert the inequality constraints to equations by introducing the non-negative ***slack*** or ***surplus*** variables. The coefficients of slack or surplus variables are always taken zero in the objective function. In this example, all inequality constraints being '≤', only slack variables s_1 and s_2 are needed. Therefore, given problem now becomes :

$$\textit{Maximize} \quad z = 3x_1 + 2x_2 + 0s_1 + 0s_2 \text{ subject to the constraints :}$$
$$x_1 + x_2 + s_1 \qquad = 4$$
$$x_1 - x_2 \qquad + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \ge 0.$$

Step 3. Now, present the constraint equations in matrix form :

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}.$$

Step 4. Construct the starting simplex table using the notations already explained *on page SMX/16* (*Table 1*)

It should be remembered that the values of non-basic variables are always zero at each iteration. So $x_1 = x_2 = 0$ here. Column $\mathbf{X_B}$ gives the values of basic variables as indicated in the first column. So $s_1 = 4$ and $s_2 = 2$ here. The complete starting basic feasible solution can be immediately read from *Table* 1 as : $s_1 = 4$, $s_2 = 2$, $x_1 = 0$, $x_2 = 0$, and the value of the objective function is *zero*.

NOTE **In this step, the variables s_1 and s_2 are corresponding to the columns of basis matrix (identity matrix), so will be called *basic variables*. Other variables, x_1 and x_2, are *non-basic variables* which always have the value zero.**

Table 1 : Starting Simplex Table

		$c_j \rightarrow$	3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3}$ ($\mathbf{S_1}$) (β_1)	$\mathbf{X_4}$ ($\mathbf{S_2}$) (β_2)	**MINIMUM RATIO** $\mathbf{X_B / X_k}$ **for** $\mathbf{X_K > 0}$
					BASIS MATRIX		
s_1	0	4	1	1	1	0	TO BE COMPUTED IN THE NEXT STEP.
s_2	0	2	1	– 1	0	1	
	$z = \mathbf{C_B X_B}$		$\Delta_1 = -3$ ↑	$\Delta_2 = -2$	$\Delta_3 = 0$	$\Delta_4 = 0$	$\leftarrow \Delta_j = z_j - \mathbf{c}_j = \mathbf{C_B X}_j - c_j$

Step 5. Now, proceed to test the basic feasible solution for optimality by the rules given below. This is done by computing the *'net evaluation'* Δ_j for each variable x_j (column vector $\mathbf{X}_j$) by the formula : $\Delta_j = z_j - \mathbf{c}_j = \mathbf{C}_B \mathbf{X}_j - c_j$ [from (7) *on page SMX/16*]

Thus, we get

$$\Delta_1 = \mathbf{C}_B \mathbf{X}_1 - c_1 = (0, 0)(1, 1) - 3 = (0 \times 1 + 0 \times 1) - 3 = -3$$

$$\Delta_2 = \mathbf{C}_B \mathbf{X}_2 - c_2 = (0, 0)(1, -1) - 2 = (0 \times 1 - 0 \times 1) - 2 = -2$$

$$\Delta_3 = \mathbf{C}_B \mathbf{X}_3 - c_3 = (0, 0)(1, 0) - 0 = (0 \times 1 + 0 \times 0) - 0 = 0$$

$$\Delta_4 = 0.$$

REMARK

It should be noted that in the starting simplex table Δ_j's are same as $(-c_j)$'s. Also, Δ_j's corresponding to the columns of unit matrix (basis matrix) are always zero. So there is no need to calculate them.

Optimality Test

(i) If all Δ_j's $(= z_j - c_j) \geq 0$, the solution under test will be ***optimal***. *Alternative optimal solutions* will exist if any non-basic Δ_j is also zero.

(ii) If at least one Δ_j is negative, the solution under test is not optimal, then proceed to improve the solution in the next step.

(iii) If corresponding to any negative Δ_j, all elements of the column $\mathbf{X}_j$ are negative or zero (≤ 0), then the solution under test will be ***unbounded.***

Applying these rules for testing the optimality of starting basic feasible solution, it is observed that Δ_1 and Δ_2 both are negative. Hence, we have to proceed to improve this solution in *Step 6.*

Step 6. In order to improve this basic feasible solution, the vector entering the basis matrix and the vector to be removed from the basis matrix are determined by the following rules. Such vectors are usually named as ***'incoming vector'*** and ***'outgoing vector'*** respectively.

'Incoming vector'. The incoming vector $\mathbf{X}_k$ is always selected corresponding to the most negative value of Δ_j (say, Δ_k). Here $\Delta_k = \min[\Delta_1, \Delta_2] = \min[-3, -2] = -3 = \Delta_1$. Therefore, $k = 1$ and hence column vector $\mathbf{X}_1$ must enter the basis matrix. The column $\mathbf{X}_1$ is marked by an upward arrow (↑).

'Outgoing vector'. The outgoing vector β_r is selected corresponding to the minimum ratio of elements of $\mathbf{X}_B$ by the corresponding *positive* elements of predetermined incoming vector $\mathbf{X}_k$. This rule is called the ***Minimum Ratio Rule.*** In mathematical form, this rule can be written as

$$\frac{x_{Br}}{x_{rk}} = \min_{i}\left[\frac{x_{Br}}{x_{ik}},\ x_{ik} > 0\right]$$

For $k = 1$,
$$\frac{x_{Br}}{x_{r1}} = \min\left[\frac{x_{B1}}{x_{11}}, \frac{x_{B2}}{x_{21}}\right] = \min\left[\frac{4}{1}, \frac{2}{1}\right] \quad \text{or} \quad \frac{x_{Br}}{x_{r1}} = \frac{2}{1} = \frac{x_{B2}}{x_{21}}.$$

Comparing both sides of this equation, we get $r = 2$. So the vector β_2, *i.e.*, $\mathbf{X}_4$ marked with downward arrow (↓) should be removed from the basis matrix. The ***Starting Simplex Table 1*** is now modified to ***Table 2*** as given below.

Table 2

	$c_j \rightarrow$		3	2	0	0	
BASIC VARIABLES	$\mathbf{C_B}$	$\mathbf{X_B}$	$\mathbf{X_1}$	$\mathbf{X_2}$	$\mathbf{X_3(S_1)}$ (β_1)	$\mathbf{X_4(S_2)}$ (β_2)	**MIN. RATIO** $\mathbf{(X_B/X_1), X_1 > 0}$
s_1	0	4	1	1	1	0	4/1
s_2	0	2	$\boxed{1}$	1	0	1	2/1← Min.
	$z = \mathbf{C_B X_B} = 0$		−3 (min. Δ_j)	−2	0	0	←$\Delta_j = z_j - c_j = \mathbf{C}_B\mathbf{X}_j - c_j$

key element

↑ entering vector

↓ leaving vector

Step 7. In order to bring $\beta_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in place of incoming vector $\mathbf{X}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, unity must occupy in the marked '□' position and zero at all other places of $\mathbf{X}_1$. If the number in the marked '□' position is other than unity, divide all elements of that row by the 'key element'. *(The element at the intersection of minimum ratio arrow (←) and incoming vector arrow (↑) is called the **key element** or **pivot element**).*

Then, subtract appropriate multiples of this new row from the other (remaining) rows, so as to obtain zeros in the remaining positions of the column $\mathbf{X}_1$. Thus, the process can be fortified by simple matrix transformation as follows :

The intermediate coefficient matrix is :

	$\mathbf{X}_B$	$\mathbf{X}_1$	$\mathbf{X}_2$	$\mathbf{X}_3$	$\mathbf{X}_4$	
R_1	4	1	1	1	0	
R_2	2	1	−1	0	1	
R_3	$z = 0$	−3	−2	0	0	← Δ_j

Apply $R_1 \rightarrow R_1 - R_2$, $R_3 \rightarrow R_3 + 3R_2$ to obtain

X_B	X_1	X_2	X_3	X_4	
2	0	2	1	−1	
2	1	−1	0	1	
$z = 6$	0	−5	0	3	$\leftarrow \Delta_j$

Now, construct the improved simplex table as follows :

Table 3

	$c_j \rightarrow$		3	2	0	0	
BASIC VARIABLES	C_B	X_B	X_1 (β_2)	X_2	$X_3(S_1)$ (β_1)	$X_4(S_2)$	**MINIMUM -RATIO** $(X_B / X_2, X_2 > 0)$
s_1	0	2	0	$\leftarrow \boxed{2}$ $\uparrow$	1	1	$\frac{2}{2} \leftarrow$ *key row*
x_1	3	2	1	−1	0	1	$\cancel{\frac{2}{-1}}$ (negative ratio is not counted)
	$z = C_B X_B = 6$		0	-5 $\uparrow$ key column	0 $\downarrow$	3	$\leftarrow \Delta_j$

From this table, the improved basic feasible solution is read as : $x_1 = 2, x_2 = 0, s_1 = 2, s_2 = 0$. The improved value of $z = 6$.

It is of particular interest to note here that Δ_j's are also computed while transforming the table by matrix method. However, the correctness of Δ_j's can be verified by computing them independently by using the formula $\Delta_j = C_B X_j - c_j$.

Step 8. Now repeat *Steps* 5 *through* 7 as and when needed until an optimum solution is obtained in *Table 4*.

$$\Delta_k = \text{most negative } \Delta_j = -5 = \Delta_2 .$$

Therefore, $k = 2$ and hence X_2 should be the entering vector (key column). By minimum ratio rule :

$$\text{Minimum Ratio} \left(\frac{X_B}{X_2}, X_2 > 0 \right) = \text{Min} \left[\frac{2}{2}, - \right]$$ (since negative ratio is not counted, so the second ratio is not considered)

Since *first ratio* is minimum, remove the first vector β_1 from the basis matrix. Hence the key element is 2. Dividing the first row by key element 2, the intermediate coefficient matrix is obtained as :

	X_B	X_1	X_2	X_3	X_4	
R_1	1	0	1	1/2	−1/2	
R_2	2	1	−1	0	1	
R_3	$z = 6$	0	−5	0	3	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 + R_1$, $R_3 \rightarrow R_3 + 5R_1$

1	0	1	1/2	−1/2	
3	1	0	1/2	1/2	
$z = 11$	0	0	5/2	1/2	$\leftarrow \Delta_j$

Now construct the next improved simplex table as follows :

Final Simplex Table 4

	$c_j \rightarrow$		3	2	0	0	
BASIC VARIABLES	C_B	X_B	$X_1 (\beta_2)$	$X_2 (\beta_1)$	S_1	S_2	
$\rightarrow x_2$	2	1	0	1	1/2	−1/2	
x_1	3	3	1	0	1/2	1/2	
	$z = C_B X_B = 11$		0	0	5/2	1/2	$\leftarrow \Delta_j$

The solution as read from this table is : $x_1 = 3, x_2 = 1, s_1 = 0, s_2 = 0$, and max. $z = 11$. Also, using the formula $\Delta_j = C_B X_j - c_j$ verify that all Δ_j's are non-negative. Hence the optimum solution is

$$x_1 = 3, x_2 = 1, \max Z = 11.$$

NOTE If at the optimal stage, it is desired to bring s_1 in the solution, the total profit will be reduced from 11 (the optimal value) to 5/2 times of 2 units of s_1 in Table 4, *i.e.*, $z = 11 - 5/2 \times 2 = 6$. This explains the *economic interpretation* of net-evaluations Δ_j.

SIMPLE WAY FOR SIMPLEX METHOD COMPUTATIONS

Complete solution with different computational steps can be more conveniently represented by the following *single* table :

Simplex Table

BASIC VARIABLES		$c_j \rightarrow$	3	2	0	0	
	C_B	X_B	X_1	X_2	S_1	S_2	MIN RATIO (X_B / X_K)
s_1	0	4	1	1	1	0	4 / 1
$\leftarrow s_2$	0	2	$\leftarrow\boxed{1}$ ↑	1	0	1	2 / 1 ← Minimum
$x_1 = x_2 = 0$	$z = C_B X_B = 0$		-3^* ↑	-2	0	0 ⊕	$\leftarrow \Delta_j = z_j - c_j$
$\leftarrow s_1$	0	2	0	$\leftarrow\boxed{2}$ ↑	1	-1	2 / 2 Min ←
$\rightarrow x_1$	3	2	1	-1	0	1	—
$x_2 = s_2 = 0$	$z = C_B X_B = 6$		0	-5^* ↑	0 ↓	3	$\leftarrow \Delta_j$
$\rightarrow x_2$	2	1	0	1	1/2	$-1/2$	
x_1	3	3	1	0	1/2	1/2	
$s_1 = s_2 = 0$	$z = C_B X_B = 11$		0	0	5/2	1/2	$\leftarrow$ All $\Delta_j \geq 0$

Thus, the optimal solution is obtained as : $x_1 = 3$, $x_2 = 1$, max $z = 11$.

Q. 1. What is a simplex ? Describe simplex method of solving linear programming problems.

2. Write the steps used in the simplex method.

3. Describe a computational procedure of the simplex method for the solution of a maximization l.p.p.

Tips for Quick Solution :

1. In the first iteration only, since Δ_j 's are the same as $- c_j$'s, so there is no need of calculating them separately by using the formula $\Delta_j = C_B X_j - c_j$.
2. Mark *min* (Δ_j) by '↑' which at once indicates the column X_k needed for computing the minimum ratio (X_B / X_k).
3. '*Key element*' is found at the place where the upward directed arrow '↑' of min Δ_j and the left directed arrow (←) of minimum ratio (X_B / X_k) intersect each other in the simplex table.
4. '*Key element*' indicates that the current table must be transformed in such a way that the key element becomes 1 and all other elements in that column become 0.
5. Since Δ_j 's corresponding to unit column vectors are always zero, so there is no need to calculate them.
6. While transforming the table by row operations, the value of z and corresponding Δ_j 's are also computed at the same time. Thus, a *lot of time and labour* can be saved in adopting this technique.

Example 2 *Min.* $z = x_1 - 3x_2 + 2x_3$ *subject to the constraints* :

$$3x_1 - x_2 + 3x_3 \leq 7, \; -2x_1 + 4x_2 \leq 12, \; -4x_1 + 3x_2 + 8x_3 \leq 10, \; and \; x_1, x_2, x_3 \geq 0.$$

[GBTU (MBA) 2011]

Solution This is the problem of minimization. Converting the objective function from minimization to maximization, we have

$$\text{Maxixize} \; -z = -x_1 + 3x_2 - 2x_3 = \text{Max.} \, z', \text{ where } -z = z'.$$

Here we give only tables of solution. The students are advised to verify them.

Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO (X_B/X_k)
$c_j \rightarrow$			−1	3	−2	0	0	0	
x_4	0	4	1	−1	1	1	1	1	—
←x_5	0	2	0	$\boxed{4}$	0	0	1	0	12/4 ¬min.
x_6	0	4	8	3	8	0	0	0	10/3
$x_1 = x_2 = x_3 = 0$	$z' = 0, z = 0$		1	−3*	2	0	0	0	←D_j
←x_4	0	10	$\boxed{5/2}$	0	3	1	1/4	0	$\frac{10}{5/2}$ ←
→x_2	0	10	−1/2	1	0	0	1/4	0	
x_6	0	1	5/2	0	8	0	−3/4	0	—
$x_1 = x_3 = x_5 = 0$	$z' = 9$ $\therefore z = -9$		−1/2*	0	2	0	−3/4	0	←D_j
←x_4	0	10	1	0	6/5	2/5	1/10	0	$\frac{10}{5/2}$ ←
→x_2	0	10	0	1	3/5	1/5	3/10	0	—
x_6	0	1	0	0	11	1	−1/2	1	
$x_3 = x_4 = x_2 = 0$	$z' = 11$ $\therefore z = -11$		0	0	13/5	1/5	8/10	0	←$D_j \geq 0$

The optimal solution is : $x_1 = 4, x_2 = 5, x_3 = 0$, *Min* $z = -11$.

Example 3 *Max.* $z = 3x_1 + 2x_2 + 5x_3$ *subject to the constraints :* **[IAS (Main) 1994]**

$$x_1 + 2x_2 + x_3 \leq 430, 3x_1 + 2x_3 \leq 460, x_1 + 4x_2 \leq 420, and\, x_1, x_2, x_3 \geq 0.$$

Solution

Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO (X_B/X_k)
$c_j \rightarrow$			3	2	5	0	0	0	
x_4	0	430	1	2	1	1	0	0	430/1
←x_5	0	460	3	0	$\boxed{2}$	0	1	0	460/2 ¬
x_6	0	420	1	4	0	0	0	1	—
$x_1 = x_2 = x_3 = 0$	$z = 0$		−3	−2	−5*	0	0	0	←D_j
←x_4	0	200	−1/2	$\boxed{2}$	0	1	−1/4	0	200/2 ←
→x_2	5	230	3/2	1	1	0	−1/2	0	
x_6	0	420	1	0	0	0	0	1	420/4
$x_1 = x_2 = x_5 = 0$	$z = 1150$		9/2	−2*	0	0	5/2	0	←D_j
←x_4	0	100	−1/4	1	0	1/2	−1/4	0	
→x_2	0	230	3/2	0	1	0	1/2	0	
x_6	0	20	2	0	0	−2	1	1	
$s_1 = s_4 = s_5 = 0$	$z = 1350$		4	0	0	1	2	0	←$D_j \geq 0$

Since all $\Delta_j \geq 0$, the solution is : $x_1 = 0, x_2 = 100, x_3 = 230$, max $z = 1350$.

Example 4 *Solve the LP problem :Max.* $z = 3x_1 + 5x_2 + 4x_3$ *subject to the constraints :*

$$2x_1 + 3x_2 \leq 8,\ 2x_2 + 5x_3 \leq 10,\ 3x_1 + 2x_2 + 4x_3 \leq 15,\ and\ x_1, x_2, x_3 \geq 0.$$

[JNTU (MBA-II Sem.) 2011, IAS (Math.) 2010]

Solution After introducing slack variables, the constraint equations become :

$$\begin{array}{lllll} 2x_1 + 3x_2 & + x_4 & & & = 8 \\ \quad 2x_2 + 5x_3 & & + x_5 & & = 10 \\ 3x_1 + 2x_2 + 4x_3 & & & + x_6 & = 15. \end{array}$$

Table 1. Starting Simplex Table

BASIC VARIABLES		$c_j \rightarrow$	3	5	4	0	0	0	
	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO $(X_B/X_2), X_2 > 0$
x_4	0	430	2	$\boxed{3}$	0	1	0	0	8/3 ←
$\leftarrow x_5$	0	460	0	2	5	0	1	0	10/2
x_6	0	420	3	2	4	0	0	1	15/2
$x_1 = x_2 = x_3 = 0$	$z = C_B X_B = 0$		−3	−5* ↑	−4	0 ↓	0	0	← Δ_j

Incoming vector outgoing vector

Now apply short-cut method for minimum ratio rule (min X_B / X_2), and find the key element 3. This key element indicates that unity should be at first place of X_2, so the vector *to be removed* from the basis matrix is X_4.

Now, in order to get the second simplex table, calculate the intermediate coefficient matrices as follows :

First, divide the first row by 3 to get

R_1	8 / 3	2 / 3	1	0	1 / 3	0	0	
R_2	10	0	2	$\boxed{5}$	0	1	0	
R_3	15	3	2	4	0	0	1	
R_4	0	− 3	− 5	− 4	0	0	0	$\leftarrow \Delta_j$

Applying $R_2 \rightarrow R_2 - 2R_1$, $R_3 \rightarrow R_3 - 2R_1$, $R_4 \rightarrow R_4 + 5R_1$,

R_1	8 / 3	2 / 3	$\boxed{1}$	0	1 / 3	0	0	
R_2	14 / 3	− 4 / 3	0	5	− 2 / 3	1	0	
R_3	29 / 3	5 / 3	0	4	− 2 / 3	0	1	
R_4	40 / 3	1 / 3	0	−4	5 / 3	0	0	$\leftarrow \Delta_j$

Now the second simplex table (Table 2) is constructed as below :

Table 2

BASIC VARIABLES		$c_j \rightarrow$	3	5	4	0	0	0	
	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO (X_B/X_3)
$\rightarrow x_4$	5	8/3	2/3	3	0	1/3	0	0	—
$\leftarrow x_5$	0	14/3	−4/3	2	$\boxed{5}$	−2/3	1	0	$\frac{14}{3}/5$ ←
x_6	0	29/3	5/3	2	4	−2/3	0	1	$\frac{29}{3}/4$
$x_4 = x_1 = x_3 = 0$	$z = \frac{40}{3}$		1/3	0	−4* ↑	5/3	0 ↓	0	← Δ_j

Incoming Outgoing

Now verify that—

$$\Delta_1 = C_B X_1 - c_1 = -3 + (5, 0, 0)\,(2/3, -4/3, 5/3) = 1/3$$

$$\Delta_4 = C_B X_4 - c_4 = 0 + (5, 0, 0)\,(1/3, -2/3, -2/3) = 5/3 .$$

$$\Delta_3 = C_B X_3 - c_3 = -4 + (5, 0, 0)\,(0, 5, 4) = -4$$

The key-element is found to be 5. Hence the vector to be removed from the basis matrix is X_5. Thus proceeding exactly in the same manner, the remaining simplex tables are obtained (Table 3 and Table 4).

Table 3

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO
x_2	5	8 / 3	2 / 3	1	0	1 / 3	0	0	$\frac{2}{3} / \frac{2}{3}$
$\rightarrow x_3$	4	14 / 15	− 4 / 15	0	1	− 2 / 15	1 / 15	0	—
$\leftarrow x_6$	0	89 / 15	$\boxed{41/15}$	0	0	− 2 / 15	− 4 / 5	1	$\frac{89}{15} / \frac{41}{15}$ ← min.
$x_1 = x_5 = x_4 = 0$	$z = 256 / 15$		− 11 / 15* ↑	0	0	17 / 15	4 / 5	0 ↓	$\leftarrow \Delta_j$

Incoming Outgoing

Table 4. Final Simplex Table

BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO
x_2	5	50 / 41	0	1	0	15 / 41	8 / 41	− 10 / 41	
x_3	4	62 / 41	0	0	1	− 6 / 41	5 / 41	4 / 41	—
x_1	3	89 / 41	1	0	0	−2 / 41	− 12 / 41	15 / 41	
$x_4 = x_5 = x_6 = 0$	$z = C_B X_B = 765/41$		0	0	0	45 / 41	24 / 41	11 / 41	$\leftarrow \Delta_j \geq 0$

Since all $\Delta_j \geq 0$, the solution given by $x_1 = 89/41, x_2 = 50/41, x_3 = 62/41$, max $z = 765/41$, is optimal.

Example 5 *Minimize* $z = x_2 - 3x_3 + 2x_5$ *subject to the constraints :*

$$3x_2 - x_3 + 2x_5 \leq 7, -2x_2 + 4x_3 \leq 12, -4x_2 + 3x_3 + 8x_5 \leq 10, \textit{and } x_2, x_3, x_5 \geq 0.$$

Solution Equivalently, max $z' = -x_2 + 3x_3 - 2x_5$ where $z' = -z$. Introducing x_1, x_4 and x_6 as slack variables, the constraint equations become :

$$x_1 + 3x_2 - x_3 + 0x_4 + 2x_5 + 0x_6 = 7$$
$$0x_1 - 2x_2 + 4x_3 + x_4 + 0x_5 + 0x_6 = 12$$
$$0x_1 - 4x_2 + 3x_3 + 0x_4 + 8x_5 + x_6 = 10.$$

Now proceeding as in above example the simplex computations are performed as follows :

Simplex Table

	$c_j \rightarrow$		0	−1	3	0	−2	0	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	X_4	X_5	X_6	MIN. RATIO $(X_B/X_k), X_k > 0$
x_1	0	7	1	3	−1	0	2	0	—
$\leftarrow x_4$	0	12	0	−2	$\leftarrow \boxed{2}$	1	0	0	$12/4 \leftarrow$
x_6	0	10	0	−4	3	0	8	1	10/3
$x_1 = x_2 = x_3 = 0$	$z' = 0$		0	1	−3*	0	2	0	$\leftarrow \Delta_j$
$\leftarrow x_1$	0	10	1	$\leftarrow \boxed{5/2}$	0	1/4	2	0	$4 \leftarrow$
$\rightarrow x_3$	3	3	0	−1/2	1	1/4	0	0	—
x_6	0	1	0	−5/2	0	−3/4	8	1	—
$x_1 = x_2 = x_5 = 0$	$z = 1150$		0	−1/2*	0	−3/4	2	0	$\leftarrow \Delta_j$
x_2	−1	4	2/5	1	0	1/10	4/5	0	
x_3	3	5	1/5	0	1	3/10	2/5	0	
x_6	0	11	1	0	0	−1/2	10	1	
	$z' = 11$ or $z = -11$		1/5	0	0	4/5	12/5	0	$\leftarrow \Delta_j \geq 0$

Thus, optimal solution is : $x_2 = 4, x_3 = 5, x_5 = 0$, min. $z = -11$.

Alternative forms of Example 5 :

(i) *Min.* $z = x_1 - 3x_2 + 2x_3$ *subject to :* $3x_1 - x_2 + 2x_3 \leq 7, -2x_1 + 4x_2 \leq 12, -4x_1 + 3x_2 + 8x_3 \leq 10$ *and* $x_1, x_2, x_3 \geq 0$.

(ii) *Min.* $z = x_2 - 3x_3 + 2x_5$ *subject to the constraints :* $x_1 + 3x_2 - x_3 + 2x_5 = 7, -2x_2 + 4x_3 + x_4 = 12, -4x_2 + 3x_3 + 8x_5 + x_6 = 10$ *and* $x_1, x_2, \ldots, x_6 \geq 0$.

Example 6 ***(Bounded Variables Problem).*** *A manufacturer of three products tries to follow a policy of producing those which continue most to fixed cost and profit. However, there is also a policy of recognising certain minimum sales requirements currently, these are :*

Product :	X_1	X_2	X_3
Units per week :	20	30	60.

There are three producing departments. The product times in hour per unit in each department and the total times available for each week in each department are :

Departments	Time required per product in hours			Total hours available
	X_1	X_2	X_3	
1	0·25	0·20	0·15	420
2	0·30	0·40	0·50	1048
3	0·25	0·30	0·25	529

The contribution per unit of product X_1, X_2, X_3 *is Rs. 10·50, Rs. 9·00 and Rs. 8·00 respectively. The company has scheduled 20 units of* X_1*, 30 units of* X_2 *and 60 units of* X_3 *for production in the following week, you are required to state :*

(i) Whether the present schedule is an optimum one from a profit point of view and if it is not, what it should be;

(ii) The recommendations that should be made to the firm about their production facilities (following the answer to (i) above).

Solution The formulation of the problem is as follows :

Maximize $z = 10{\cdot}5X_1 + 9X_2 + 8X_3$, *subject to the constraints :*

$$0{\cdot}25X_1 + 0{\cdot}20X_2 + 0{\cdot}15X_3 \le 420$$
$$0{\cdot}30X_1 + 0{\cdot}40X_2 + 0{\cdot}50X_3 \le 1048$$
$$0{\cdot}25X_1 + 0{\cdot}30X_2 + 0{\cdot}25X_3 \le 529$$
$$0 \le X_1 \ge 20 , 0 \le X_2 \ge 30 , 0 \le X_3 \ge 60 .$$

Since the company is already producting minimum of X_2 and X_3, it should, at least, produce maximum of X_1 limited by the first constraint. Lower bounds are specified in this problem, *i.e.*, $X_1 \ge 20$, $X_2 \ge 30$, $X_3 \ge 60$. This can be handled quite easily by introducing the new variables x_1, x_2 and x_3 such that

$$X_1 = 20 + x_1, X_2 = 30 + x_2, X_3 = 60 + x_3 .$$

Substituting for X_1, X_2 and X_3 in terms of x_1, x_2, x_3, the problem now becomes :

Maximize $z = 10{\cdot}5x_1 + 9x_2 + 8x_3$ + constant, *subject to the constraints :* $0{\cdot}25x_1 + 0{\cdot}20x_2 + 0{\cdot}15x_3 \le 400$, $0{\cdot}30x_1 + 0{\cdot}40x_2 + 0{\cdot}50x_3 \le 1000$, $0{\cdot}25x_1 + 0{\cdot}30x_2 + 0{\cdot}25x_3 \le 500$, *and* $x_1 \ge 0, x_2 \ge 0, x_3 \ge 0$.

The students may now proceed to find the optimal solution by simplex method in the usual manner.

Example 7 ***(Product Mix Problem)*** *For a company engaged in the manufacture of three products, viz. X, Y and Z, the available data are given below :*

Minimum Sales Requirement

Product :	X	Y	Z
Min. sales requirement per month :	10	20	30

Operations, Required Processing Times and Capacity

Operations	Time (hrs.) required per item of			Total available hours per month
	X	Y	Z	
1	1	2	2	200
2	2	1	1	220
3	3	1	2	180

Profit (Rs.) per unit

Product :	X	Y	Z
Profit (Rs.)/unit :	10	15	8

Find out the product-mix to maximize profit.

Solution Let x, y and z denote the number of units produced per month for the products X, Y and Z, respectively.

Minimum sales requirements give the constraints : $x \ge 10$, $y \ge 20$, $z \ge 30$, where $x, y, z \ge 0$.

Operations, processing times and capacity lead to the following constraints :

$$x + 2y + 2z \le 200 \text{ ...(i)} \qquad 2x + y + z \le 220 \text{ ..(ii)} \qquad 3x + y + 2z \le 180 \text{ ...(iii)}$$

The objective function is : Max. $\mathbf{P} = 10x + 15y + 8z$. Thus we have to solve the following problem :

Max. $\mathbf{P} = 10x + 15y + 8z$, *subject to* $x + 2y + 2z \le 200$, $2x + y + z \le 220$, $3x + y + 2z \le 180$, and $0 \le x \ge 10$, $0 \le y \ge 20$, $0 \le z \ge 30$.

Let us make the substitutions : $x = a + 10$, $y = b + 20$, $z = c + 30$, where $a, b, c \ge 0$.

Substituting these values in the objective function and constraints (i), (ii) and (iii), the problem becomes :

Max. $\mathbf{P} = 10a + 15b + 8c + 640$, *subject to,*

$$(a + 10) + 2(b + 20) + 2(c + 30) \le 200$$
$$2(a + 10) + (b + 20) + (c + 30) \le 220$$

$$3(a+10)+(b+20)+2(c+30) \le 180,$$

where $a \ge 0, b \ge 0, c \ge 0$.

Solving this problem by simplex method we get the solution : $a = 10$, $b = 40$ and $c = 0$. Substituting these values, we find the original values :

$x = 10 + 10 = 20$, $y = 40 + 20 = 60$, $z = 0 + 30 = 30$, and the maximum value of objective function is given by $P = \text{Rs.}1340$.

The optimal product mix is to produce 20 units of X, 60 units of Y, and 30 units of Z to get a maximum profit of Rs. 1340.

Example 8 *Nooh's Boats makes three different kinds of boats. All can be made profitably in this company, but the company's monthly production is constrained by the limited amount of labour, wood and screws available each month. The director will choose the combination of boats that maximizes his revenue in view of the information given in the following table :*

Input	Row Boat	Canoe	Keyak	Monthly Available
Labour (Hours)	12	7	9	1.260 hrs.
Wood (Board feet)	22	18	16	19,008 board feet
Screws (Kg.)	2	4	3	396 Kg.
Selling price (in Rs.)	4,000	2,000	5,000	

(a) Formulate the above as a linear programming problem.

(b) Solve it by simplex method. From the optimal table of the solved linear programming problem, answer the following questions :

(c) How many boats of each type will be produced and what will be the resulting revenue ?

(d) Which, if any, of the resources are not fully utilized ? If so, how much of spare capacity is left ?

(e) How much wood will be used to make all of the boats given in the optimal solution ?

Solution (a) Let x_1, x_2 and x_3 be the number of *Row Boats, Canoe* and *Keyak* made every month. The linear programming model can be formulated as follows :

Max. Revenue $z = 4{,}000\,x_1 + 2{,}000\,x_2 + 5{,}000\,x_3$, *subject to*

$$12\,x_1 + 7\,x_2 + 9\,x_3 \le 1260,\ 22\,x_1 + 18\,x_2 + 16\,x_3 \le 19008,\ 2\,x_1 + 4\,x_2 + 3\,x_3 \le 396 \text{ and } x_1, x_2, x_3, \ge 0.$$

(b) Adding slack variables s_1, s_2, s_3, the above formulated problem becomes

Max. $z = 4000x_1 + 2000x_2 + 5000x_3 + 0s_1 + 0s_2 + 0s_3$, *subject to :*

$$12\,x_1 + 7\,x_2 + 9\,x_3 + s_1 = 1260,\ 22\,x_1 + 18\,x_2 + 16\,x_3 + s_2 = 19008,\ 2\,x_1 + 4\,x_2 + 3\,x_3 + s_3 = 396, \text{ and}$$

$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0.$$

The starting solution and subsequent simplex tables are given below :

		$c_j \rightarrow$	4000	2000	5000	0	0	0	
BASIC VARIABLES	**Prog. C_B**	**Qty X_B**	**X_1**	**X_2**	**X_3**	**S_1**	**S_2**	**S_3**	**Replacement Ratio min (X_B/X_K)**
s_1	0	1,260	12	7	9	1	0	0	1260/9
s_2	0	19,008	22	18	16 ↑	0	1	0	19008/16
s_3	0	396	2	4	$\leftarrow\boxed{3}$	0	0	1	396/3 ←
	$z' = 0$		−4000	−2000	−5000 ↑	0	0	0 ↓	← Δ_j (NER)
s_1	0	72	$\leftarrow\boxed{6}$	−5	0	1	0	−3	12 ←
s_2	0	16.896	34/3	−10/3	0	0	1	−16/3	1491
x_3	5000	132	2/3	4/3	1	0	0	1/3	198
	$z = 660000$		−2000/3	−14,000/3 ↑	0	0 ↓	0	5000/3	← Δ_j
x_1	4000	12	1	−5/6	0	1/6	0	−1/2	
s_2	0	16,760	0	55/9	0	−17/6	1	1/3	
x_3	5000	124	0	17/9	1	−1/9	0	2/3	
	$z = 6{,}68{,}000$		0	37,000/9	0	1000/9	0	4000/3	← $\Delta_j \ge 0$

Since all $\Delta_j \ge 0$, the optimal solution is given by $x_1 = 12$, $x_2 = 0$ and $x_3 = 124$.

(c) The company should produce 12 *Row boats* and *124 Kayak boats* only. The maximum revenue will be Rs. 6,68,000.

(d) Wood is not fully utilized. Its share capacity is 16,760 board feet.

(e) The total wood used to make all of the boats given by the optimum solution is

$$= 22 \times 12 + 16 \times 124 = 2{,}248 \text{ board feet.}$$

SUMMARY OF COMPUTATIONAL PROCEDURE OF SIMPLEX METHOD

Simplex method is an iterative procedure involving the following steps :

Step 1. If the problem is one of minimization, convert it to a maximization problem by considering $-z$, instead of z, using the fact min $z = -\max(-z)$ or $\min z = -\max.(z'), z' = -z$.

Step 2. We check up all b_i' s for nonnegativity. If some of the b_i' s are negative, multiply the corresponding constraints through by -1 in order to ensure all $b_i \geq 0$.

Step 3. We change the inequalities to equations by adding *slack* and *surplus* variables, if necessary.

Step 4. We add *artificial variables* to those constraints with ($\geq$) or (=) sign in order to get the identity basis matrix.

Step 5. We now construct the starting simplex table (*see Table below* for all problems). From this table, the initial basic feasible solution can be read off.

Form of Simplex Table

		$c_j \rightarrow$	c_1	c_2	c_3	...	c_k	...	c_{m+n}	
BASIC VARIABLES	C_B	X_B	X_1	X_2	X_3	...	X_k	...	X_{m+n}	**MIN. RATIO RULE**
...	...	...	...	...	...	...	...	...	...	...
	$z = C_B X_B$		Δ_1	Δ_2	Δ_3	...	Δ_k	...	Δ_{m+n}	$\leftarrow \Delta_j$

Step 6. We obtain the values of Δ_j by the formula $\Delta_j = z_j - c_j = C_B X_j - c_j$, and examine the values of Δ_j. There will be three mutually exclusive and collectively exhaustive possibilities :

(i) All $\Delta_j \geq 0$. In this case, the basic feasible solution under test will be optimal.

(ii) Some $\Delta_j < 0$, and for at least one of the corresponding $\mathbf{X}_j$ all $x_{rj} \leq 0$. In this case, the solution will be unbounded.

(iii) Some $\Delta_j \leq 0$, and all the corresponding $\mathbf{X}_j$'s have at least one $x_{ij} > 0$. In this case, there is no end of the road. So further improvement is possible.

Step 7. Further improvement is done by replacing one of the vectors at present in the basis matrix by that one out side the basis. We use the following rules to select such a vector :

(i) *To select "incoming vector". We find such value of k* for which $\Delta_k = \min \Delta_j$. Then the vector coming into the basis matrix will be $\mathbf{X}_k$.

(ii) *To Select "outgoing vector". The vector going out of the basis matrix will be* β_r, *if we determine the suffix r* by the minimum ratio rule

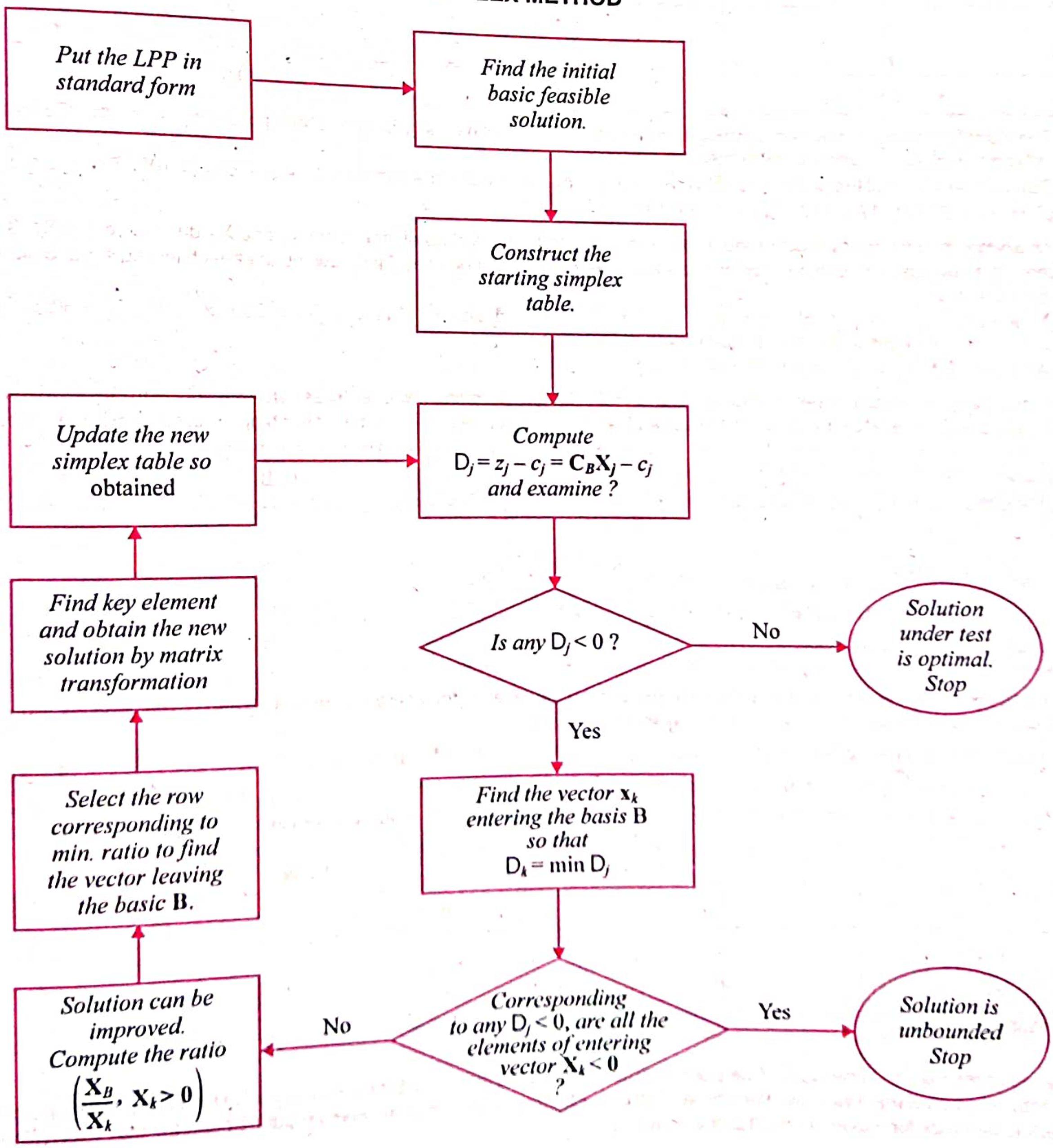
$$\frac{x_{Br}}{x_{rk}} = \text{for predetermined value of } k.$$

Step 8. We now construct the next improvement table by using the simple matrix transformation rules.

Step 9. Now return to ***Step 6,*** then go the ***steps 8*** and ***9,*** if necessary. This process is repeated till we reach the desired conclusion.

NOTE All the steps of simplex algorithm can be easily remembered by the Flow-Chart given below

FLOWCHART FOR SIMPLEX METHOD



CHAPTER XII

Sensitivity Analysis

12.1 INTRODUCTION.

In a practical situation of solving linear programming problem the elements of the co-efficient matrix **A**, the components of the requirement vector **b** and the cost vector **c**, called the *input parameters,* are neither known exactly nor they are constant for a model for a specified period. Thus it is important to know how sensitive the optimal solution is to small discrete changes in these parameters. By sensitiveness we mean fulfilment of the condition of optimality as well as determining the limits of variations of these parameters for the solution to remain optimal feasible.

We shall study the effect of changes in the

(i) co-efficients (c_j) of the objective function,

(These co-efficients may be of the basic variables or of the non-basic variables)

(ii) components of the requirement vector (**b**),

(iii) co-efficients (a_{ij}) of the variables of the constraints.

(These co-efficients may be of the basic variables or of the non-basic variables.)

There is another type of modification in the linear programming model in which

(ix) a new constraint is added to the set of original constraints or

(x) a new variable becomes necessary to the original formulation. This happens if the original formulation be unfortunately found to be erroneous or because the model situation is changed. These changes may result in the change in the optimal basic solution or in their values or both.

The discussion of how sensitive a given optimal solution is, as a result of various discrete changes in the input parameters, is usually called *sensitivity analysis*. This together with the investigation of how changes in the input parameters affect the optimal solution is called *post optimal analysis*. Sensitivity analysis reduces the additional computational effort which arises in solving the problem anew.

12.2 CHANGES IN THE OBJECTIVE FUNCTION.

When changes are made in the objective function only, the optimal solution is still feasible as the feasible region remains unaltered.

Let us consider the change in the objective function when the cost c_k becomes $(c_k + \delta_k)$. This cost co-efficient may be associated with the non-basic variable or it may be associated with a basic variable in the optimal solution.

(i) *When* c_k *is associated with the non-basic variable* x_k.

Let $\mathbf{x}_B^*$ be an optimal basic feasible solution to the *L. P. P.*

$$\text{Maximize } z = \mathbf{cx}$$

$$\text{subject to } \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}$$

and let **B** be the optimal basis matrix.

Let $\mathbf{c}_B$ be the cost vector corresponding to $\mathbf{x}_B^*$. For the optimal solution, $z_j - c_j \geq 0$ for all j. If δ_k be the change in c_k, then $\mathbf{c}_B$ is not changed as c_k is not in $\mathbf{c}_B$. As there is no change in **B**, $z_j - c_j = \mathbf{c}_B\, \mathbf{y}_j - c_j$ will remain unchanged and non-negative for the basic vectors $\mathbf{a}_j$ corresponding to the current optimal solution. Only $(z_k - c_k)$ will change, where c_k corresponds to the non-basic vector $\mathbf{a}_k$. Hence, for all j, the inequality $z_j - c_j \geq 0$ will hold, if $z_k - (c_k + \delta_k) \geq 0$. Hence the optimal solution to remain optimal. . . . (1)

$$\delta_k \leq z_k - c_k$$

If the cost c_k of any non-basic variable x_k be increased by more than the amount $(z_k - c_k)$, then the resulting $(z_k - c_k)$ will be negative and a few more iterations will be necessary to determine the new optimal solution.

Note that the cost of any non-basic variable can be reduced without limit, without affecting the optimality of $\mathbf{x}_B^*$.

(ii) *When c_k is associated with a basic variable x_k.*

As c_k is changed to $(c_k + \delta_k)$, let the optimal basis cost $\mathbf{c}_B$ be changed to $\mathbf{c}'_B$. $\mathbf{c}_B$ being changed, all $(z_j - c_j)$ will also be changed for all j corresponding to which the vectors are not in the optimal basis. For the vectors in the optimal basis, $z_j - c_j = 0$. Let z_j be changed to z_j' corresponding to their change in c_k. The solution obtained will remain optimal (maximal), if $z_j' - c_j \geq 0$ for all j corresponding to non-basis vectors.

Now let $\mathbf{c}_B = (c_1, c_2, \ldots\ldots, c_k, \ldots\ldots, c_m)$

so that $\mathbf{c}_B' = (c_1, c_2, \ldots\ldots, c_k + \delta_k, \ldots\ldots, c_m)$

$$= (c_1, c_2, \ldots\ldots, c_k, \ldots\ldots, c_m) + (0, 0, \ldots\ldots, \delta_k, \ldots\ldots, 0)$$

$$= \mathbf{c}_B + \delta_k \mathbf{I}_k,$$

$\mathbf{I}_k$ being the unit vector with 1 as the k-th component.

Now we know $\mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_j = z_j$ and $\mathbf{B}^{-1}\mathbf{a}_j = \mathbf{y}_j$; therefore

$$z_j' = \mathbf{c}_B' \mathbf{B}^{-1}\mathbf{a}_j = (\mathbf{c}_B + \delta_k \mathbf{I}_k)\mathbf{B}^{-1}\mathbf{a}_j$$

$$= \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_j + \delta_k \mathbf{I}_k \mathbf{y}_j$$

$$= z_j + \delta_k y_{kj}, \text{ with } \mathbf{I}_k \text{ as defined earlier.}$$

Thus the condition $z'_j - c_j \geq 0$ is equivalent to $z_j + \delta_k y_{kj} - c_j \geq 0$, for all j not in the optimal basis. This is the same as to say

$$-(z_j - c_j) \leq \delta_k y_{kj}.$$

Hence the obtained solution will remain optimal (maximal) feasible, if

$$-\frac{(z_j - c_j)}{y_{kj}} \leq \delta_k, \text{ for } y_{kj} > 0$$

and

$$-\frac{(z_j - c_j)}{y_{kj}} \geq \delta_k, \text{ for } y_{kj} < 0.$$

These two can be combined to write

$$\text{Min}\left(-\frac{z_j - c_j}{y_{kj}},\ y_{kj} < 0\right) \geq \delta_k \geq \text{Max}\left(-\frac{z_j - c_j}{y_{kj}},\ y_{kj} > 0\right) \quad \ldots(2)$$

for all j, for which $\mathbf{a}_j$ is not in the basis.

If δ_k lies in the range as given earlier, then the solution remains optimal. If δ_k falls outside this range, then at least one $(z_j - c_j)$ will be negative and the solution will no longer remain optimal.

If no $y_{kj} > 0$, then there is no lower bound of δ_k and if no $y_{kj} < 0$, then there is no upper bound of δ_k.

12.3 VARIATIONS IN THE REQUIREMENT VECTOR.

We know that $z_j - c_j = \mathbf{c}_B \mathbf{B}^{-1}\mathbf{a}_j - c_j$. So the factor determining the optimality condition does not depend on **b**, the requirement vector. Hence a change in the vector **b** by an amount **d** (positive or negative) does not change the optimality of the solution. All that we are to check is the feasibility of the new solution, as the solution $\mathbf{x}_B = \mathbf{B}^{-1}\mathbf{b}$ will be changed by changing **b**.

Consider that we have found a solution $\mathbf{x}_B^*$ of the linear programming problem

$$\text{Maximize} \quad z = \mathbf{cx},$$
$$\text{subject to} \quad \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{0}.$$

Then let the i-th component of **b** be changed by an amount d_i (positive or negative) so that the new i-th component becomes

$$\bar{b}_i = b_i + d_i \ (i = 1, 2, \ldots\ldots, m),$$

that is, in vector notation $\bar{\mathbf{b}} = \mathbf{b} + \mathbf{d}$, where

$$\mathbf{d} = (0, 0, \ldots\ldots, d_i, 0, \ldots\ldots, 0).$$

If now corresponding to the optimal basis **B** of the original problem, the changed problem has a solution with the same basis **B** represented by $\bar{\mathbf{x}}_B$, then

$$\bar{\mathbf{x}}_B = \mathbf{B}^{-1}\bar{\mathbf{b}} = \mathbf{B}^{-1}(\mathbf{b} + \mathbf{d}) = \mathbf{B}^{-1}\mathbf{b} + \mathbf{B}^{-1}\mathbf{d} = \mathbf{x}_B^* + \mathbf{B}^{-1}\mathbf{d}.$$

Hence feasibility of the new solution will depend on $\mathbf{B}^{-1}\mathbf{d}$. As $(z_j - c_j)$ is independent of the requirement vector, $\bar{\mathbf{x}}_B$ will be optimal, if it be feasible. Now if $\bar{\mathbf{x}}_B$ be not feasible, then one or more $\bar{x}_{B_i}$ will be negative and we can use the dual simplex method to find the new optimal solution.

Now $\mathbf{B}^{-1}$ is necessary to compute $\overline{\mathbf{x}}_B$ from the final tableau obtained by applying the revised simplex method or the simplex method after we perform all computations on the columns which contain the identity matrix used to obtain the initial basic feasible solution. If now the columns of **A** be so arranged that the first m of its columns constitute the basis matrix, then

$$\mathbf{B}^{-1} = [\,\mathbf{y}_1, \mathbf{y}_2, \mathbf{y}_3, \ldots\ldots, \mathbf{y}_m\,]$$

and $\overline{\mathbf{x}}_B = \mathbf{x}_B^* + \mathbf{B}^{-1}\mathbf{d} = \mathbf{x}_B^* + \sum_{j=1}^{m} d_j\, \mathbf{y}_j$

that is, $\overline{x}_{B_i} = x_{B_i} + \sum_{j=1}^{m} d_j\, y_{ij}$.

When only one component of **b**, say b_k, is changed, then the i-th basic variable of the new problem is given by

$$\overline{x}_{B_i} = x_{B_i} + y_{ik}\, d_k$$

where y_{ik} is the $(\,i, k\,)$-th element of $\mathbf{B}^{-1}$.

For the feasibility of the new solution, we must have

$$x_{Bi} + y_{ik}\ d_k \geq 0$$

or, $d_k \geq -\dfrac{x_{Bi}}{y_{ik}}$, if $y_{ik} > 0$

and $d_k \leq -\dfrac{x_{Bi}}{y_{ik}}$, if $y_{ik} < 0$.

Thus if we choose d_k, the change in b_k, such that

$$\underset{y_{ik} > 0}{\text{Max}} \left\{ -\frac{x_{Bi}}{y_{ik}} \right\} \leq d_k \leq \underset{y_{ik} < 0}{\text{Min}} \left\{ -\frac{x_{Bi}}{y_{ik}} \right\},$$

then the current optimal solution will remain feasible.

For the change in the value of the objective function, suppose z is changed to $\overline{z}$ when b_k is changed to $(\,b_k + d_k\,)$.

Then $\overline{z} = \mathbf{c}_B\, \mathbf{B}^{-1}\, \overline{\mathbf{b}} = \mathbf{c}_B\, \overline{\mathbf{x}}_B$.

12.4 CHANGES IN THE CO-EFFICIENT MATRIX.

If one or more of the elements of the co-efficient matrix **A** be changed, then the problem becomes complicated. If we change an element from a column of **A** which is a basic vector in the optimal solution, then the optimal basis matrix is recomputed ; as a result the m columns of **A** corresponding to the current basic variables may remain linearly independent or may not be independent. Even if they be linearly independent, new $\mathbf{B}^{-1}$ as well as $\mathbf{y}_j$ and $(z_j - c_j)$ are to be computed afresh.

(i) Suppose we replace the vector $\mathbf{a}_k$, a non-basic vector of A, by making changes in one or more elements as given by

$$\bar{\mathbf{a}}_k = \mathbf{a}_k + \alpha$$

where the vector α is of m components. As $\mathbf{a}_k$ is non-basic, the optimal basis matrix **B** remains unchanged and hence the quantities

$$\mathbf{x}_B^* = \mathbf{B}^{-1}\mathbf{b},\ \mathbf{c}_B \text{ and } z_j - c_j = \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_j - c_j \quad \text{for } j \neq k$$

remain unchanged. The only change will be in $\mathbf{y}_k$ given by

$$\bar{\mathbf{y}}_k = \mathbf{B}^{-1}\bar{\mathbf{a}}_k = \mathbf{B}^{-1}(\mathbf{a}_k + \alpha) = \mathbf{B}^{-1}\mathbf{a}_k + \mathbf{B}^{-1}\alpha$$

$$= \mathbf{y}_k + \mathbf{B}^{-1}\alpha.$$

Again in this case

$$\bar{z}_k - c_k = \mathbf{c}_B\mathbf{B}^{-1}\bar{\mathbf{a}}_k - c_k = \mathbf{c}_B\mathbf{B}^{-1}(\mathbf{a}_k + \alpha) - c_k$$

$$= \mathbf{c}_B\mathbf{B}^{-1}\mathbf{a}_k - c_k + \mathbf{c}_B\mathbf{B}^{-1}\alpha$$

$$= z_k - c_k + \mathbf{c}_B\mathbf{B}^{-1}\alpha.$$

Now if $\bar{z}_k - c_k \geq 0$, the present optimal solution remains optimal ; but if $\bar{z}_k - c_k < 0$, then $\mathbf{y}_k$ is to be computed and the optimal solution is to be found through few more iterations.

For a single change δ_{lk} in $\mathbf{a}_k$ in the l-th element, we have the new element $\quad a_{lk} + \delta_{lk}$

and hence for the optimality of this solution, we must have

$$\mathbf{c}_B\mathbf{B}^{-1}(\mathbf{a}_k + \delta_{lk}\mathbf{I}_l) - c_k \geq 0$$

where $\mathbf{I}_l$ is the unit vector having 1 in the l-th position.

This condition implies

$$z_k - c_k + \delta_{lk} \cdot \sum_{l=1}^{m} y_{il} \, c_{Bi} \geq 0,$$

where y_{il} is the element of $\mathbf{B}^{-1}$ in the i-th row and l-th column.

Hence optimality and feasibility will be maintained, if

$$\underset{i}{\text{Min}} \; \frac{-(z_k - c_k)}{\left(\sum_{i=1}^{m} y_{il} \, c_{Bi}\right) < 0} \geq \delta_{lk} \geq \underset{i}{\text{Max}} \; \frac{-(z_k - c_k)}{\left(\sum_{i=1}^{m} y_{il} \, c_{Bi}\right) > 0}.$$

For a non-existent denominator the corresponding bound will not exist.

(ii) If the change in elements be desired in a vector $\mathbf{a}_k$ which is basic, we shall first remove this vector from the basis before giving any change to the elements of $\mathbf{a}_k$ and then we shall make the desired change in the presently obtained non-basic vector $\mathbf{a}_k$.

In this case we can otherwise resolve the problem from the beginning or recompute $\mathbf{B}^{-1}$, all $\mathbf{y}_j$ and $(z_j - c_j)$.

12.5 ADDITION OF A VARIABLE.

If it be seen that somehow a variable has been left out of the formulation of the problem and if it be desired to add a new variable, say x_{n+1}, with the associated activity vector $\mathbf{a}_{n+1}$ and having the price c_{n+1} to the problem, then we can restate the linear programming problem as follows :

$$\text{Maximize } z = \mathbf{c}\mathbf{x} + c_{n+1} \, x_{n+1}$$

$$\text{subject to} \quad [\mathbf{A}, \; \mathbf{a}_{n+1}] \begin{bmatrix} \mathbf{x} \\ x_{n+1} \end{bmatrix} = \mathbf{b},$$

$$\mathbf{x} \geq \mathbf{0}, \; x_{n+1} \geq 0.$$

If now $\mathbf{x}_B^*$ be the optimal solution to the original problem, then it will be a basic feasible solution to the newly constructed problem with the new variable x_{n+1} being zero, since it is currently non-basic. But this solution will be optimal for the new problem, if

$$z_{n+1} - c_{n+1} \geq 0, \text{ for } z_j - c_j \geq 0 \text{ for all } j = 1, 2, \ldots\ldots, n,$$

and these are not changed by the addition of a new variable which is non-basic.

Now computing $z_{n+1} - c_{n+1} = \mathbf{c}_B \mathbf{B}^{-1} \mathbf{a}_{n+1} - c_{n+1}$ if it be found to be negative, then we are to calculate $\mathbf{y}_{n+1} = \mathbf{B}^{-1} \mathbf{a}_{n+1}$ and proceed with the simplex or the revised simplex method so that $\mathbf{a}_{n+1}$ is brought in the basis.

12.6 ADDITION OF A CONSTRAINT.

The addition of a new constraint to the constraint set of the given problem means the addition of a new variable (slack or surplus), associated with the new constraint.

Let the constraint set be $\mathbf{A}\mathbf{x} = \mathbf{b}, \mathbf{x} \geq \mathbf{0}$. Let us add a new constraint to this in the form

$$\sum_{j=1}^{m} a_{n+1,j}\, x_j + x_{n+1} = b_{m+1} \qquad \ldots \quad (1)$$

where x_{n+1} is a slack or surplus variable (hence $c_{n+1} = 0$) added to the new constraint. Thus addition of a new constraint means addition of a new variable. b_{m+1} is not necessarily positive as the added constraint may be either of the "less than" or "greater than" type. If the current optimal solution satisfies (1), then it is still the optimal solution since the objective function is not changed thereby ; otherwise we shall have to find the new optimal solution for the new set of ($m+1$) equations consisting of ($m+1$) basic vectors.

12.7 ILLUSTRATIVE EXAMPLES.

Ex. 1. *Find the limits of variations of the costs c_1, c_2, c_3, c_4, c_5 and c_6 respectively for the linear programming problem whose optimal table is given below, so that the optimal solution remains optimal.*

			c_j	-1	-1	3	0	-3	0
$\mathbf{c}_B$	**B**	$\mathbf{x}_B$	**b**	$\mathbf{a}_1$	$\mathbf{a}_2$	$\mathbf{a}_3$	$\mathbf{a}_4$	$\mathbf{a}_5$	$\mathbf{a}_6$
-1	$\mathbf{a}_2$	x_2	5	$\frac{2}{5}$	1	0	$\frac{1}{10}$	$\frac{4}{5}$	0
3	$\mathbf{a}_3$	x_3	6	$\frac{1}{5}$	0	1	$\frac{3}{10}$	$\frac{2}{5}$	0
0	$\mathbf{a}_6$	x_6	8	1	0	0	$-\frac{1}{2}$	10	1
			$z_j - c_j$	$\frac{6}{5}$	0	0	$\frac{4}{5}$	$\frac{17}{5}$	0

From the tableau we see that the optimal solution is

$$x_1 = 0,\ x_2 = 5,\ x_3 = 6,\ x_4 = x_5 = 0,\ x_6 = 8 \text{ and } z_{max} = 13.$$

We notice that x_1, x_4, x_5 are the non-basic variables in the optimal tableau. Hence the permissible limits of the variations of c_1, c_4, c_5 which are δ_1, δ_4, δ_5 respectively, so that the above solution remains optimal, are given by

$$\delta_1 \leq z_1 - c_1, \quad i.e., \quad \delta_1 \leq \tfrac{6}{5},$$

$$\delta_4 \leq z_4 - c_4, \quad i.e., \quad \delta_4 \leq \tfrac{4}{5},$$

$$\delta_5 \leq z_5 - c_5, \quad i.e., \quad \delta_5 \leq \tfrac{17}{5}.$$

Now we find the permissible limits for the variations of c_2, c_3, c_6, which are the costs corresponding to the basic variables x_2, x_3, x_6. By the formula we have, for δ_2, that is, the variation of c_2

$$\text{Min}\left(-\frac{z_j - c_j}{y_{2j}},\ y_{2j} < 0\right) \geq \delta_2 \geq \text{Max}\left(-\frac{z_j - c_j}{y_{2j}},\ y_{2j} > 0\right)$$

for all j corresponding to non-basis vectors. From the tableau we see that

$$y_{2j} = \left(\frac{2}{5},\ 1,\ 0,\ \frac{1}{10},\ \frac{4}{5},\ 0\right)$$

for, these are the elements of the second row corresponding to the variable x_2 associated to which the cost is c_2. We are to consider those y_{2j} for which j corresponds to the non-basis vectors $(j = 1, 4, 5)$, that is, $(y_{21},\ y_{24},\ y_{25})$, that is, $\left(\frac{2}{5},\ \frac{1}{10},\ \frac{4}{5}\right)$.

Similarly, for δ_3, we consider (corresponding to c_3)

$$(y_{31},\ y_{34},\ y_{35}), \text{ that is, } \left(\frac{1}{5},\ \frac{3}{10},\ \frac{2}{5}\right)$$

and for δ_6, we consider (corresponding to c_6)

$$(y_{61},\ y_{64},\ y_{65}), \text{ that is, } \left(1,\ -\frac{1}{2},\ 10\right)$$

Then applying the criterion we have

$$\infty \text{ (as no } y_{2j} < 0) \geq \delta_2 \geq \text{Max}\left(\frac{-\frac{6}{5}}{\frac{2}{5}}, \frac{-\frac{4}{5}}{\frac{1}{10}}, \frac{-\frac{17}{5}}{\frac{4}{5}}\right)$$

or, $\infty \geq \delta_2 \geq -3.$

$$\text{Similarly, } \infty \geq \delta_3 \geq \text{Max}\left(\frac{-\frac{6}{5}}{\frac{1}{5}}, \frac{-\frac{4}{5}}{\frac{3}{10}}, \frac{-\frac{17}{5}}{\frac{2}{5}}\right)$$

or, $\infty \geq \delta_3 \geq -\frac{8}{3}$

$$\text{and Min}\left(-\frac{\frac{4}{5}}{-\frac{1}{2}}\right) \geq \delta_6 \geq \text{Max}\left(-\frac{\frac{6}{5}}{1}, -\frac{\frac{17}{5}}{10}\right)$$

or, $\frac{8}{5} \geq \delta_6 \geq -\frac{6}{5}.$

Variations in c_2, c_3, c_6 by amounts specified by δ_2, δ_3, δ_6 as given above will not disturb the optimality of the present solution.

Ex. 2. *Find the optimal solution of the L. P. P.*

Maximize $z = 4x_1 + 3x_2$

subject to $x_1 + x_2 \leq 5,$

$3x_1 + x_2 \leq 7,$

$x_1 + 2x_2 \leq 10,$

$x_1, x_2 \geq 0.$

Show how to find the optimal solution of the problem, if

(i) the first component of the original requirement vector be increased by one unit and the third component be decreased by one unit ;

(ii) the second component of the original requirement vector be decreased by two units.

[*Calcutta M. Sc., 1985 ; Vidyasagar M. Sc., 1995*]

(i) We add the slack variables x_3, x_4 and x_5 to the constraints and apply simplex method to solve the problem. The initial basis matrix is given by the columns $\mathbf{a}_3$, $\mathbf{a}_4$, $\mathbf{a}_5$. Easily we get the optimal solution from the table as given in the next page.

			c_j	4	3	0	0	0
c_B	B	x_B	b	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
3	$\mathbf{a}_2$	x_2	4	0	1	$\frac{3}{2}$	$-\frac{1}{2}$	0
4	$\mathbf{a}_1$	x_1	1	1	0	$-\frac{1}{2}$	$\frac{1}{2}$	0
0	$\mathbf{a}_5$	x_5	1	0	0	$-\frac{5}{2}$	$\frac{1}{2}$	1
			$z_j - c_j$	0	0	$\frac{5}{2}$	$\frac{1}{2}$	0

The optimal basis inverse, that is, $\mathbf{B}^{-1} = [\mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5]$, for these columns afforded the initial basis.

$$\therefore\ \mathbf{B}^{-1} = [\mathbf{y}_3, \mathbf{y}_4, \mathbf{y}_5] = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix}.$$

Now the requirement vector **b** becomes (**b** + **d**), where **d** is given by $\mathbf{d} = (1, 0, -1)$.

Thus $\bar{x}_B = \mathbf{x}_B^* + \mathbf{B}^{-1}\mathbf{d}$

$$= \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$= \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} \\ -\frac{1}{2} \\ -\frac{7}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{2} \\ \frac{1}{2} \\ -\frac{5}{2} \end{bmatrix}.$$

Thus we see that the new optimal solution is not feasible as $x_1 = \frac{1}{2}$, $x_2 = \frac{11}{2}$ and $x_5 = -\frac{5}{2}$ (slack variable). Hence we are to take the help of the dual simplex method to find the new feasible solution.

We see that $\mathbf{a}_3$ will enter the basis and $\mathbf{a}_5$ will leave it.

The new table is

			c_j	4	3	0	0	0
c_B	B	x_B	b	y_1	y_2	y_3	y_4	y_5
3	a_2	x_2	4	0	1	0	$-\frac{1}{5}$	$\frac{3}{5}$
4	a_1	x_1	1	1	0	0	$\frac{2}{5}$	$-\frac{1}{5}$
0	a_3	x_3	1	0	0	1	$-\frac{1}{5}$	$-\frac{2}{5}$
			$z_j - c_j$	0	0	0	1	1

The new optimal solution is thus $x_1 = 1$, $x_2 = 4$ and $z_{max} = 16$.

(ii) Here $\mathbf{d} = (0, -2, 0)$.

$$\therefore \quad \bar{\mathbf{x}}_B = \mathbf{x}_B^* + \mathbf{B}^{-1}\mathbf{d} = \begin{bmatrix} 4 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 \\ -\frac{5}{2} & \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 0 \end{bmatrix}.$$

Hence the optimal solution is $x_1 = 0$, $x_2 = 5$; $z_{max} = 15$.

Note. It should be noted that a change in only one component of the requirement vector may change all the components of the optimal solution.

Ex. 3. *The optimal simplex table for a given L. P. P. is given below :*

			c	4	3	4	6	0	0	0
c_B	B	x_B	b	y_1	y_2	y_3	y_4	y_5	y_6	y_7
6	a_4	x_4	$\frac{18}{13}$	0	$\frac{14}{13}$	0	1	$\frac{4}{13}$	$-\frac{5}{13}$	$\frac{2}{13}$
4	a_3	x_3	$\frac{20}{13}$	0	$-\frac{23}{13}$	1	0	$-\frac{1}{13}$	$\frac{11}{13}$	$-\frac{7}{13}$
4	a_1	x_1	$\frac{28}{13}$	1	$\frac{16}{13}$	0	0	$-\frac{1}{13}$	$-\frac{2}{13}$	$\frac{6}{13}$
			$z_j - c_j$	0	$\frac{17}{13}$	0	0	$\frac{16}{13}$	$\frac{6}{13}$	$\frac{8}{13}$

The given problem is a maximizing problem with all the constraints "$\leq$" type.

Determine the separate ranges of discrete changes in a_{12}, a_{22} and a_{32} consistent with the optimal solution of the given problem.

[Calcutta M. Sc., 1982]

It is obvious from the optimal table that x_5, x_6, x_7 are the slack variables associated with the given constraints of the problem and hence [$\mathbf{a}_5$, $\mathbf{a}_6$, $\mathbf{a}_7$] gives the initial basis matrix.

If B be the optimal basis, then

$$\mathrm{B}^{-1} = \begin{bmatrix} \frac{4}{13} & -\frac{5}{13} & \frac{2}{13} \\ -\frac{1}{13} & \frac{11}{13} & -\frac{7}{13} \\ -\frac{1}{13} & -\frac{2}{13} & \frac{6}{13} \end{bmatrix} = [\,\delta_1\,,\ \delta_2\,,\ \delta_3\,],\ \text{(say)}.$$

Now $\mathbf{c}_B\delta_1 = 6\left(\frac{4}{13}\right) + 4\left(-\frac{1}{13}\right) + 4\left(-\frac{1}{13}\right) = \frac{16}{13}$,

$\mathbf{c}_B\delta_2 = 6\left(-\frac{5}{13}\right) + 4\left(\frac{11}{13}\right) + 4\left(-\frac{2}{13}\right) = \frac{6}{13}$

and $\mathbf{c}_B\delta_3 = 6\left(\frac{2}{13}\right) + 4\left(-\frac{7}{13}\right) + 4\left(\frac{6}{13}\right) = \frac{8}{13}$.

Now a_{12}, a_{22}, a_{32} are the elements of the vector $\mathbf{a}_2$ which is non-basic as is evident from the optimal table. Thus a change in $\mathbf{a}_2$ will only violate the condition of optimality. Hence the range of discrete changes in the coefficients of the vector $\mathbf{a}_2$ for the optimality of the present solution is given by $\left(z_2 - c_2 = \frac{17}{13} \right)$

$\dfrac{-\frac{17}{13}}{\frac{16}{13}} \le$ change in a_{12} , that is, change in $a_{12} \ge -\frac{17}{16}$,

$\dfrac{-\frac{17}{13}}{\frac{6}{13}} \le$ change in a_{22} , that is, change in $a_{22} \ge -\frac{17}{6}$,

$\dfrac{-\frac{17}{13}}{\frac{8}{13}} \le$ change in $\dot{a}_{32}$, that is, change in $a_{32} \ge -\frac{17}{8}$.

Ex. 4. *The optimum simplex table for a maximization problem (with all constraints "≤" type) is*

c_B	B	b	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$
12	$\mathbf{a}_2$	$\frac{8}{5}$	0	1	$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$
5	$\mathbf{a}_1$	$\frac{9}{5}$	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$
			0	0	$\frac{17}{5}$	$\frac{29}{5}$	$M-\frac{2}{5}$

where x_4 is the slack and x_5 is the artificial variable. Let a new variable $x_6 \geq 0$ be introduced in the problem with a cost 18 assigned to it in the objective function. Suppose that the new vector corresponding to the variable x_6 be [3, 2]. Discuss the effect of this addition of a variable on the optimality of the optimal solution of the given problem.

x_4 and x_5 being the slack and the artificial variables, $[\mathbf{a}_4, \mathbf{a}_5]$ constituted the initial basis. If **B** be the optimal basis, then

$$\mathbf{B}^{-1} = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

Let $\mathbf{a}_6$ be the vector corresponding to the variable x_6.

$$\therefore \quad \mathbf{y}_6 = \mathbf{B}^{-1}\mathbf{a}_6 = \begin{bmatrix} \frac{2}{5} & -\frac{1}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{4}{5} \\ \frac{7}{5} \end{bmatrix}.$$

The cost vector $\mathbf{c}_B = [12, 5]$ as seen from the optimal table and $c_3 = 6$.

$$\text{Now } z_6 - c_6 = \mathbf{c}_B\,\mathbf{y}_6 - c_6 = [12, 5] \begin{bmatrix} \frac{4}{5} \\ \frac{7}{5} \end{bmatrix} - 18 = -\tfrac{7}{5} < 0.$$

Thus the optimality condition is violated and the new simplex table is

$\mathbf{c}_B$	**B**	**b**	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
12	$\mathbf{a}_2$	$\frac{8}{5}$	0	1	$\frac{1}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	$\frac{4}{5}$
5	$\mathbf{a}_1$	$\frac{9}{5}$	1	0	$\frac{7}{5}$	$\frac{1}{5}$	$\frac{2}{5}$	$\frac{7}{5}$
			0	0	$\frac{17}{5}$	$\frac{29}{5}$	$M - \frac{2}{5}$	$-\frac{7}{5}$

We see that $\mathbf{a}_6$ enters the basis and $\mathbf{a}_1$ leaves the basis.

The new simplex table is

c_B	**B**	**b**	$\mathbf{y}_1$	$\mathbf{y}_2$	$\mathbf{y}_3$	$\mathbf{y}_4$	$\mathbf{y}_5$	$\mathbf{y}_6$
12	$\mathbf{a}_2$	$\frac{4}{7}$	$-\frac{4}{7}$	1	$-\frac{3}{5}$	$\frac{2}{7}$	$-\frac{3}{7}$	0
18	$\mathbf{a}_6$	$\frac{9}{7}$	$\frac{5}{7}$	0	1	$\frac{1}{7}$	$\frac{2}{7}$	1
			1	0	$\frac{24}{5}$	6	M	0

Since all $z_j - c_j \geq 0$, the optimal solution is obtained.

Hence the new optimal basic feasible solution is

$$x_1 = 0,\ x_2 = \frac{4}{7} \text{ and } z_{max} = \frac{48}{7}.$$

Previous maximum value of z was $\frac{141}{5}$.